MARCH 1952 NUMBER 3

PROCEEDINGS

OF THE

NATIONAL ACADEMY OF SCIENCES

OF THE

UNITED STATES OF AMERICA

EXECUTIVE COMMITTEE

OF THE PROCEEDINGS:

Roger Bidwell Wilcox, *Managing Editor*

R. A. Brink D. W. Brogl

Publication Office: Mack Printing Company, Easton, Pa.

Editorial Office: Harvard School of Public Health, Boston 15

Home Office of the Academy, Washington 25, D. C.

Entered as second class matter May 31, 1938, at the post office at Easton, Pa., under the act of August 24, 1912.
Accepted for mailing at special postage provided for in section 1103, act of October 3, 1917. Authorized June 28, 1938.

Price per annum, $5.00

OFFICERS OF THE ACADEMY

PRESIDENT ..... A. H. KIRKLAND
VICE PRESIDENT ..... ROGER B. WHEAT
FOREIGN SECRETARY ..... D. W. BRONK
HOME SECRETARY ..... PAUL R. WEIDLEIN
TREASURER ..... W. J. ROBBINS

EDITORIAL BOARD

THE HOME SECRETARY and the FOREIGN SECRETARY of the ACADEMY
THE CHAIRMAN of the NATIONAL RESEARCH COUNCIL

A. ERVIN	E. W. GOODENOUGH	W. V. OSTERHOUT
R. E. CLELAND	F. B. JEWETT	P. H. SELLERS
A. L. DAY	A. B. LAMB	C. H. SWEETSER
C. A. ELTRINGHAM	E. C. LAWRENCE	S. A. STEVENS
L. P. EISENHART	W. R. MAGEE	L. B. WILSON
R. C. GIBBS	W. A. NOYES, JR.	S. WRIGHT

INFORMATION TO SUBSCRIBERS

Checks covering subscriptions to the PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES at the rate of $6.00 per annum should be made payable to the National Academy of Sciences and sent to the TREASURER, National Academy of Sciences, 2101 Constitution Ave., Washington 25, D. C. Single numbers $0.50.

Past publications of the NATIONAL ACADEMY OF SCIENCES from 1863 to 1926 and of the NATIONAL RESEARCH COUNCIL from 1919 to 1925 are listed in the *Index to the First Ten Volumes of these PROCEEDINGS*. This Index appeared as Part 2 of VOLUME 13, No. 1, January, 1927, and is for sale separately at the price of $1.00. Orders should be sent to the Home Secretary, National Academy of Sciences, 2101 Constitution Ave., Washington 25, D. C.

Past volumes of the PROCEEDINGS may be obtained at five dollars per volume unless the sale of the volume would break a complete set of Volumes I to XXX. Single numbers may be obtained for fifty cents except when the sale of such numbers would break up a complete volume. Only sixty complete sets of the PROCEEDINGS are available for sale—Volumes I to XXX price $300.00. Orders should be sent to the Home Secretary, National Academy of Sciences, 2101 Constitution Ave., Washington 25, D. C.

The following publications are issued by the NATIONAL RESEARCH COUNCIL. Orders and inquiries should be addressed: Publication Office, National Research Council, 2101 Constitution Ave., Washington 25, D. C.

(A) THE BULLETIN OF THE NATIONAL RESEARCH COUNCIL, published at irregular intervals. Individual numbers priced variously. A discount of 20% from the list price will be given to all who subscribe for the series as issued, direct from the Publication Office.

(B) THE REPRINT AND CIRCULAR SERIES OF THE NATIONAL RESEARCH COUNCIL, individual numbers variously priced. A discount of 20% from the list price will be given to all who subscribe for the series as issued direct from the Publication Office.

PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES

Volume 36 March 15, 1950 Number 3

GALACTIC AND EXTRAGALACTIC STUDIES, XX. ON THE DISTRIBUTION OF 78,000 OF THE BRIGHTER NORTHERN GALAXIES

By Harlow Shapley

Harvard College Observatory

Communicated January 3, 1950

In several earlier papers in this series the distribution and brightness of galaxies with magnitudes between 13 and 18 have been reported. Because of the relation of magnitude to distance, these surveys have involved space distribution as well as surface arrangement.

In the present communication the surface distribution alone is given for nearly 80,000 objects in the northern galactic hemisphere, north of declination +41°. Since only a small fraction of the objects marked on the photographic plates is fainter than magnitude 18, the extragalactic systems here reported can be referred to as the brighter galaxies. The distribution fainter than magnitude 18, and more distant for the average galaxy than 140,000,000 light years, will be a project for several of the newer telescopes that are now or soon will be undertaking explorations deep into the metagalaxy.

1. Scope of the Survey.—The 153 plates on which this survey depends were made with the Metcalf refractor at the Oak Ridge Station by Henry A. Sawyer, using Cramer Hi-Speed Special (blue) plates all with exposures of three hours. Miss Rebecca Jones examined most of the plates and tabulated the nebular counts; Mrs. Nail has assisted in the numerical discussions.

Since the main object of this prolonged survey of northern galaxies is, first, to examine the true clustering and intrinsic distributional irregularities (as distinguished from the uneven blocking in low latitudes by interstellar obscuring clouds), and, secondly, to evaluate the relation of galactic latitude to the frequency of seventeenth magnitude systems, no fields nearer the Milky Way than galactic latitude +20° are included. This limitation has also the advantage of minimizing the misidentifications near the plate

limit that occur in the richer star fields of low latitude. It is believed that the survey is reasonably accurate, and that the number of faint double or multiple stars erroneously included (estimated at about three per cent of the entries) is balanced, on the average, by the over-looked galaxies, which are generally of the spheroidal type. The degree of accuracy of the identifications has been sample-checked with the aid of the large reflectors.

The plates show stars to the eighteenth magnitude (median working limit is 17.9), and the galaxies are therefore recorded with essential completeness to magnitude 17.5. All galaxies within a hundred million light years, with absolute magnitudes brighter than -15, are thus included, but the dwarfs at that distance are too faint for inclusion.

The region of the sky covered includes the whole of the constellations Draco and Ursa Minor, most of Lynx and Ursa Major and considerable portions of Canes Venatici, Boötes, Hercules and Camelopardalis. It may be of interest that 1500 galaxies are shown on these Harvard plates in the bowl of the Big Dipper. If the radial density is approximately constant, there should be in this area, bounded by stars of the Dipper's bowl, more than one million galaxies photographable with long exposures on fast plates with the Hale reflector on Palomar.

2. *Summary of Observations.*—In table 1 the observational material is summarized, with plate centers given in both equatorial and galactic coordinates. The magnitude limits, m_s, refer to stars. The galaxy census is shown in five columns. N_T refers to the whole plate (about 35 square degrees); N_{25} to the central area of 25 square degrees, which encompasses for each plate practically all that can be used without application of distance corrections to the magnitudes and to the counted numbers of galaxies. N_9 gives the total population for the central 9 square degrees, and the last two columns the data per square degree reduced to a common magnitude limit of 17.9.

3. *The Magnitude Limits.*—The equatorial and galactic coordinates in table 1 give the center of the plates within a tenth of a degree. After the plates were marked for galaxies and the magnitude limit for stars determined, the galaxies were counted with the aid of a superposed glass reticule with squares one degree on the side. There seems to be no point in recording the positions of these numerous and generally featureless extragalactic objects more closely than is here provided. The actual numbers in each square degree for each plate will be published in the *Harvard Annals* and thus permit a detailed analysis of the distribution.

The varying sky conditions and emulsion speeds have produced the differing plate limits (sixth column). The limits have been determined for each plate through the star-count method, which is based on fundamental Mount Wilson magnitudes in Selected Areas. In this northern part of the sky, away from the congestions and irregularities of the Milky Way, star

TABLE 1

POSITIONS AND COUNTS

PLATE	α (1900)	δ	λ	β	m_z	N_T	N_m	N_3	REDUCED N_m	REDUCED N_3
27801*	$0^h\ 5^m.1$	+87° 19′	90°	+25°	17.6	112	94	25	5.7	4.2
28001*	2 2.0	83 05	93	22	18.0	141	120	34	4.2	3.3
28034*	4 7.6	83 06	96	24	17.8	105	79	25	3.7	3.2
28048*	4 18.4	77 24	101	20	18.2	173	141	36	3.7	2.6
29142	5 42.3	68 25	112	21	18.0	181	152	45	5.3	4.2
28054*	5 53.8	78 00	103	25	17.9	308	242	65	9.7	7.2
27640‡	5 55.2	72 37	109	23	18.1	378	260	74	7.9	6.2
28049*	6 23.9	82 11	99	27	18.0	411	333	98	11.6	9.5
29487	6 27.2	52 56	130	20	18.5	255	215	104	3.8	5.1
29810	6 27.6	58 11	125	22	18.3	161	126	47	2.9	3.0
29264	6 34.8	63 03	120	24	18.1	490	370	126	11.2	10.6
29216	6 40.7	67 41	115	26	18.2	523	388	111	10.2	8.1
29931	6 53.5	48 17	136	23	18.2	784	646	209	17.0	15.3
28089	6 55.6	87 11	93	28	17.7	136	123	54	6.5	7.9
29781	6 57.6	52 53	131	24	18.0	387	297	77	10.3	7.5
28733†	7 5.0	44 00	141	23	18.3	570	468	170	10.7	10.9
27521*	7 7.8	72 34	110	29	18.2	491	392	137	10.4	10.0
29213	7 8.9	58 13	126	27	18.2	619	506	140	13.3	10.3
29853	7 22.9	62 41	121	30	18.1	584	427	154	13.0	13.0
28666*	7 24.7	77 02	105	30	18.3	431	345	93	7.9	5.9
29944	7 32.2	67 46	115	31	18.1	316	261	80	7.9	6.2
29310	7 33.8	48 23	138	29	18.0	662	478	116	16.6	11.2
28753†	7 36.8	44 01	143	29	18.2	571	453	153	11.9	11.7
30052	7 46.4	52 59	133	32	18.4	944	722	306	14.4	17.0
29306	7 57.0	58 03	127	33	17.7	697	610	274	32.2	40.1
28603	8 6.3	48 35	138	35	17.9	593	478	191	19.1	21.2
29948	8 11.0	62 47	121	35	18.0	887	672	267	23.4	25.8
29493†	8 15.0	43 39	145	35	18.0	711	583	185	20.3	17.9
26903‡	8 21.2	72 38	109	34	18.5	605	506	204	8.8	9.9
30037	8 21.7	53 16	133	36	17.9	726	593	217	23.7	24.1
28261*	8 28.7	82 35	98	31	18.1	234	183	67	5.5	5.6
31364	8 38.1	58 29	126	39	17.8	369	319	108	14.7	13.8
29215†	8 38.5	44 01	144	40	18.2	926	716	214	18.9	15.7
28686	8 39.3	67 47	114	37	17.7	650	527	182	27.8	26.7
30040	8 44.6	48 54	138	41	18.1	1342	1012	394	30.8	33.3
30651	9 0.2	53 21	132	43	17.8	703	587	206	27.0	26.3
28699	9 2.5	62 49	119	41	17.8	363	292	116	13.4	14.8
28667*	9 5.7	77 28	102	35	18.2	540	448	160	11.8	11.7
27687†	9 17.9	43 23	145	47	18.4	835	663	198	13.2	11.0
29257	9 22.7	58 33	124	45	17.8	354	284	110	13.1	14.0
29877	9 24.2	48 15	138	47	17.9	1135	849	324	34.0	36.0
31367	9 31.6	67 38	112	41	17.6	406	314	130	18.9	21.7
28239*	9 33.9	72 42	106	39	18.1	482	385	128	11.7	10.8
31220	9 44.8	53 37	129	49	17.7	883	660	222	34.8	32.6
30078†	9 47.6	43 47	144	52	18.5	1226	1028	380	18.0	18.4
27701	9 50.8	62 57	116	46	18.0	607	490	157	17.1	15.1

TABLE 1 (*Continued*)

PLATE	α	(1900)	b		λ	β	m_g	N_T	$N_{\mathfrak{M}}$	N_0	REDUCED $\bar{N}_{\mathfrak{M}}$	REDUCED $\bar{N}_0$
29932	9	58.1	49	04	134	53	18.0	985	766	232	26.6	22.4
28865	10	6.8	58	28	120	50	18.2	1084	819	295	21.6	21.6
29278	10	19.8	53	07	126	54	17.9	1361	1069	377	42.8	41.9
29265†	10	22.1	43	29	142	58	18.4	1377	1096	420	21.9	23.3
30903	10	27.6	68	00	107	45	17.7	595	507	159	26.8	23.4
30740	10	31.2	48	47	131	58	18.1	2396	1792	612	54.5	51.7
28161*	10	34.8	77	57	98	38	17.9	518	453	164	18.1	18.2
28735	10	36.9	62	42	111	50	17.6	404	322	127	19.5	21.3
28056*	10	38.3	72	49	102	42	17.9	699	557	235	22.3	26.1
27584‡	10	50.2	58	00	115	54	17.9	815	650	216	26.0	24.0
29283†	10	54.8	43	27	137	64	18.5	1016	811	273	14.2	13.2
28868	10	55.8	53	02	120	58	18.3	820	641	185	14.7	11.8
28078*	11	4.0	82	17	94	35	17.9	333	264	89	10.6	9.9
31292	11	6.2	48	29	125	63	17.6	739	575	192	34.7	32.2
29524†	11	15.3	72	16	99	44	17.9	724	579	232	23.2	25.8
28788	11	25.2	43	43	129	68	18.1	769	594	216	18.1	18.2
29308	11	28.4	68	08	100	49	17.7	551	465	172	24.5	25.2
28109	11	32.1	63	15	103	53	17.7	1033	799	309	42.2	45.3
28719	11	35.0	58	32	106	57	18.1	1289	1035	319	31.5	26.9
28754	11	40.9	48	19	115	67	17.8	1320	1072	399	49.3	50.9
27641‡	11	42.0	53	50	108	62	18.5	2273	1837	614	32.1	29.8
28057*	11	56.0	87	32	90	30	18.0	224	162	53	5.6	5.1
27639†‡	11	58.1	43	47	115	72	18.3	1277	974	371	22.4	23.7
28086*	12	0.5	77	27	93	40	17.6	314	258	90	15.6	15.1
28095*	12	11.1	73	06	93	45	17.7	568	470	161	24.8	23.6
28938	12	17.7	48	52	100	69	18.0	985	797	279	27.7	27.0
30797	12	18.3	53	45	97	64	17.8	1463	1006	366	49.0	46.8
28913	12	18.8	63	35	94	54	17.8	1085	907	346	41.7	44.2
27688	12	19.5	58	19	95	59	17.9	893	757	262	30.3	29.1
27704†	12	27.6	43	45	98	74	18.1	878	677	206	20.6	17.4
30125	12	35.2	67	57	91	50	17.8	858	681	256	31.3	32.7
31359	12	50.6	48	40	85	69	17.8	779	588	218	27.0	27.8
27635*	12	54.0	82	41	90	35	18.1	555	447	136	13.6	11.5
30851	12	55.4	53	24	85	64	17.8	1469	1207	394	55.5	50.4
30038†	13	04.1	43	42	74	74	18.0	1435	1172	447	40.8	43.2
30041	13	6.4	58	32	83	59	17.7	773	620	210	32.7	30.8
27194*	13	10.8	72	04	87	46	18.1	558	447	209	13.6	17.6
28213	13	13.0	62	58	83	55	17.6	510	395	144	23.9	24.2
27645*	13	23.6	77	25	87	40	18.0	973	831	321	28.9	1.1
29377	13	24.9	48	21	70	68	17.8	578	441	161	20.2	20.6
30742	13	26.4	67	32	83	50	17.8	808	581	187	26.7	23.9
30837	13	35.9	53	24	72	63	17.6	641	491	169	29.6	28.4
30129†	13	41.8	43	26	55	70	18.1	1079	890	329	27.1	27.8
29498	13	49.9	58	24	73	57	17.8	1154	890	283	40.9	36.1
30053	13	58.6	63	01	75	53	17.6	669	526	164	31.7	27.5
29447	14	04.2	48	40	58	64	17.9	723	569	179	22.8	19.9

TABLE 1 (*Continued*)

PLATE	α (1900)	δ	λ	β	m_r	N_T	N_m	N_s	REDUCED $\overline{N_m}$	REDUCED $\overline{N_s}$
28914	14 13.4	53 43	63	59	17.8	742	566	194	26.0	24.7
29962†	14 13.4	43 59	47	65	18.3	714	583	233	13.4	14.9
30132	14 18.4	68 12	77	47	17.6	723	551	198	33.5	33.2
28158*	14 32.1	72 55	79	43	17.7	354	298	108	15.7	15.8
29606	14 36.8	49 08	52	59	17.8	747	567	163	26.1	20.8
28947	14 38.3	58 22	64	53	17.4	561	441	158	35.2	35.2
29979†	14 42.8	43 51	41	61	18.1	639	474	142	14.4	12.0
29559	14 48.9	63 11	68	49	17.4	691	582	228	46.6	50.6
28172*	14 53.5	77 48	82	38	17.7	425	363	128	19.1	18.7
30927	14 54.3	53 44	56	55	17.5	739	560	215	39.0	41.6
27088*	15 2.7	84 20	86	33	17.8	511	411	157	18.9	20.0
28949†	15 10.5	43 25	37	56	18.0	550	415	127	14.4	12.3
30929	15 13.6	48 26	45	54	17.5	523	430	172	29.9	33.2
29504	15 16.5	58 50	60	49	17.5	401	337	133	23.5	25.7
30880	15 19.4	68 8	71	44	17.5	780	622	240	43.3	46.5
27373*	15 29.1	72 54	75	40	17.9	384	312	90	12.5	10.0
32917	15 31.2	53 21	51	50	18.0	824	672	236	23.4	22.8
32932	15 44.6	48 7	42	49	18.1	834	668	260	20.3	22.0
27406	15 45.9	62 54	63	44	18.0	649	488	182	17.0	17.6
30079†	15 48.9	44 00	36	49	18.2	1726	1425	462	37.6	33.8
27462*	15 55.4	75 05	76	37	17.8	399	281	105	12.9	13.5
28945	16 00.1	58 50	56	44	17.5	518	421	160	29.2	31.0
27192*	16 4.4	79 19	80	34	17.9	623	470	161	18.8	17.9
27273*	16 8.9	87 44	88	29	17.5	274	223	94	15.5	18.1
29556	16 13.4	53 28	49	44	17.6	597	481	171	29.0	28.7
30886	16 21.4	68 05	67	38	17.8	959	712	202	32.8	25.8
31445	16 23.3	48 28	42	43	17.6	386	321	86	19.3	14.5
30186†	16 25.3	43 34	35	43	18.0	1143	939	310	32.7	29.9
29529	16 31.6	63 04	60	39	17.3	611	475	149	43.5	37.8
27230*	16 46.3	71 55	71	35	18.2	770	659	204	17.4	15.0
28940	16 46.3	57 39	53	38	17.5	435	358	107	24.9	20.7
31418	16 51.9	53 11	47	38	17.7	602	465	185	24.6	27.2
29537†	16 54.4	43 48	35	38	17.9	500	367	109	14.7	12.1
31441	16 57.8	48 24	41	37	17.7	537	402	125	21.2	18.3
29539	17 15.5	67 29	65	34	17.5	501	366	111	25.4	21.4
28302	17 22.8	62 50	59	33	17.8	356	273	104	12.5	13.2
30123†	17 26.6	43 29	36	32	18.3	808	616	236	14.1	15.1
27320	17 29.3	57 56	53	33	17.9	481	416	144	16.6	16.0
29564	17 30.1	52 39	47	32	17.8	503	400	145	18.4	18.5
29567	17 36.6	48 29	42	31	17.6	465	370	148	22.3	24.8
27279*	17 58.0	77 22	76	30	17.8	605	504	172	23.2	22.0
30224†	17 58.1	43 32	38	26	17.7	442	373	153	19.7	22.4
27288*	18 1.6	72 24	70	29	17.8	548	440	153	20.2	19.6
27445*	18 3.4	83 26	83	29	18.2	447	355	103	9.4	7.5
28939	18 5.9	62 56	59	29	17.5	315	260	92	18.1	17.6
32320	18 9.4	48 12	43	25	17.6	155	131	54	7.9	9.1

TABLE 1 (*Continued*)

PLATE	α (1900)		δ		λ	β	m_g	N_T	N_m	N_r	REDUCED $\bar{N}_m$	$\bar{N}_r$
29618	18	10.3	67	52	65	29	17.7	332	237	90	12.5	13.2
32975	18	13.4	57	54	54	27	17.9	421	379	171	15.2	19.0
28948	18	14.0	53	15	49	26	17.7	211	160	55	8.4	8.0
28950†	18	33.9	43	08	39	20	18.0	222	189	69	6.6	6.7
32464	18	45.9	48	34	45	20	17.3	114	92	28	8.5	7.1
32469	18	49.6	52	47	50	21	17.3	208	158	51	14.4	13.0
30939	18	56.3	58	03	55	21	17.7	259	186	51	9.8	7.5
27431	19	01.7	62	32	60	22	17.8	335	280	76	12.9	9.7
28478*	19	11.4	72	04	70	24	17.8	330	286	97	13.1	12.4
28328	19	12.6	67	28	66	23	18.1	292	256	99	7.7	8.4
27293*	19	27.7	77	41	77	25	18.0	225	182	76	6.4	7.3
27753*	19	37.2	82	32	82	26	18.2	255	210	68	5.5	5.0
27796*	21	8.4	77	42	80	20	17.7	62	47	13	2.5	1.8
27808*	22	2.6	82	22	85	22	17.8	61	52	15	2.4	1.9
27837*	23	57.3	82	22	89	20	17.8	61	50	16	2.3	2.1

* Shapley and Jones, *Harv. Ann.*, **106** No. 1 (1938).

† Shapley and Jones, these PROCEEDINGS, **26** 554–561 (1940), Harvard Reprint 208.

‡ Seyfert, *Harv. Ann.*, **105** No. 10 (1937).

counts provide a reliable procedure. The magnitude limits for the 153 plates are distributed as follows:

Magnitude limit	17.3	17.4	17.5	17.6	17.7	17.8	17.9
Number of fields	3	2	9	13	18	28	17
Reduction factor	2.29	2.00	1.74	1.51	1.32	1.15	0.00
Magnitude limit	18.0	18.1	18.2	18.3	18.4	18.5	
Number of fields	18	17	13	7	3	5	
Reduction factor	0.87	0.76	0.66	0.575	0.50	0.437	

The reduction factor in the last line is based on the customary assumption, which the Harvard and Mount Wilson studies have both supported, that the average space density of galaxies in high latitudes is essentially uniform, at least out to a distance of 10^8 light years. The reduction factor has been applied only for the last two columns of table 1, and for figure 1 and table 3.

4. *Test of Metcalf Telescope Field.*—The curved glass plates so accurately fit the field of the Metcalf doublet that no appreciable distance error is found within a circle of $2^1/_2°$ radius. There may be, however, slight inequalities in the field, as is shown by the data of table 2, which are based on the nebular counts on the 153 plates. (No correction of any kind has been applied to the data in this tabulation.) In each square are the number of the square, the mean population of galaxies $\bar{N}$ for that square, the deviation $\bar{N}-20.4$, and the mean error ϵ of $\bar{N}$ (based on 153 fields). For the whole plate $\bar{N}$ averages 20.4 $\pm$ 0.2 (m. e., based on 25 squares), and ϵ averages $\pm 0.51 \pm 0.01$ (m. e.).

The deviations $\bar{N}-20.4$ provide a test of the uniformity of the field of the telescope. In a sense, we are here using some 80,000 galaxies to test, through the deviations from the value 20.4 galaxies per square degree, the performance of a 4-lens objective. The test is not absolute, of course, because irregularities of nebular distribution and observational uncertainties also enter, and are not wholly smoothed out by the large number of plates. The deviations from 20.4 are for 16 of the squares less than twice the corresponding mean error ϵ, and therefore show no sensible distortion of the telescopic field at those points. Only for squares 1, 3, 14, 22, 23 and 24 does the evidence suggest that the effectiveness is definitely more or less

TABLE 2

TEST OF THE TELESCOPE'S FIELD

(See text, section 2, for description)

25	24	23	22	21
20.6	21.9	22.4	23.3	19.7
+0.2	+1.5	+2.0	+2.9	−0.7
±0.57	±0.60	±0.56	±0.60	±0.49
20	19	18	17	16
20.0	19.5	19.2	20.1	19.2
−0.4	−0.9	−1.2	−0.3	−1.2
±0.52	±0.48	±0.40	±0.44	±0.56
15	14	13	12	11
20.2	19.0	19.3	20.6	19.6
−0.2	−1.4	−1.1	+0.2	−0.8
±0.50	±0.52	±0.40	±0.48	±0.51
10	9	8	7	6
20.5	19.5	19.8	21.4	20.8
+0.1	−0.9	−0.6	+1.0	+0.4
±0.49	±0.49	±0.45	±0.49	±0.50
5	4	3	2	1
19.6	21.1	23.0	21.4	18.2
−0.8	+0.7	+2.6	+1.0	−2.2
±0.44	±0.53	±0.58	±0.55	±0.56

than the average. If rich clusters of galaxies should accidentally fall on the same square for several plates, a large positive average deviation might result. If vacancies fortuitously combine on a square, a large negative average deviation could occur. To test further this possibility of clustering, the fields of table 1 were separated into three equal groups, in order of number. The deviations are positive and large in all three groups for square 22 only; and for square 14 only, large and negative (deficiency of recorded galaxies). We conclude that the inequalities of table 2 are primarily due to clustering and fortuitous "holes," and not to instrumental causes.

The deviations of table 2 are a very sensitive measure of the telescope's field. The largest are about 12 per cent of the mean nebular numbers for

those squares. On the usual uniform-density assumption, this percentage corresponds to a deviation of only 0.08 mag. in effective reach. The average is 5 per cent, corresponding to 0.04 mag. In practice, therefore, for nebular surveys, the field inequalities are negligible. If they exist, they probably arise from differences in sharpness of focus; either the lenses or the curved-plate matrix may be at fault. With smooth fields of stars rather than patchy nebular fields, this testing method for large field lenses or mirrors could be made very precise.

5. Distribution of Galaxies in Latitude and Longitude.—A comparison of the results obtained from the study of the 3600 square degrees, reported in the present communication, with other areas surveyed in the southern sky, will be made in the course of a forthcoming general summary of the Harvard work on the distribution of galaxies. The summary will deal with the distribution of the brighter systems for the whole sky, and for a large fraction of the sky will concern the more distant objects down to magnitude 18, of which well over half a million have been photographed with the Harvard telescopes, but less than 200,000 studied in detail.

TABLE 3

MEAN NUMBER OF GALAXIES PER SQUARE DEGREE

GALACTIC LATITUDE	LONGITUDE INTERVALS				
	30–59°	60–89°	90–119°	120–149°	30–149°
20–29°	12.1 (9)	9.2 (12)	7.2 (11)	11.1 (10)	9.7 (42)
30–39°	18.8 (9)	22.6 (10)	11.6 (10)	21.1 (7)	18.3 (36)
40–49°	28.0 (6)	26.0 (9)	20.7 (9)	24.5 (7)	24.5 (31)
50–59°	26.6 (5)	31.0 (7)	31.8 (7)	28.6 (7)	29.7 (26)
>60°	19.4 (4)	34.6 (5)	33.5 (6)	22.3 (3)	28.8 (18)
>40°	25.2 (15)	29.7 (21)	27.7 (22)	25.8 (17)	27.4 (75)
All	19.9 (33)	22.3 (43)	18.7 (43)	20.5 (34)	20.4 (153)

Two results from the earlier analysis of a part of the material of table 1, both of significance in the cosmography of external galaxies, are confirmed by the present larger study. The first is the demonstration of the far extension of the "Cepheus flare" or cloud of absorbing material that comes out of the Milky Way between galactic longitudes 70° and 110° and extends up to galactic latitude 37° in longitude 105°. This flare of absorption covers the North Celestial Pole, and as pointed out in earlier reports, indicates that the North Polar magnitude standard sequence is somewhat obscured. The color as well as the brightness of the standard stars must be affected. This absorbing cloud is shown in latitudes higher than +20° in the density diagram in figure 1, where each circular area is centered on a plate center and covers 20 square degrees. The actual coverage of the sky is more complete than the diagram indicates, since the plates actually cover 35 square degrees.

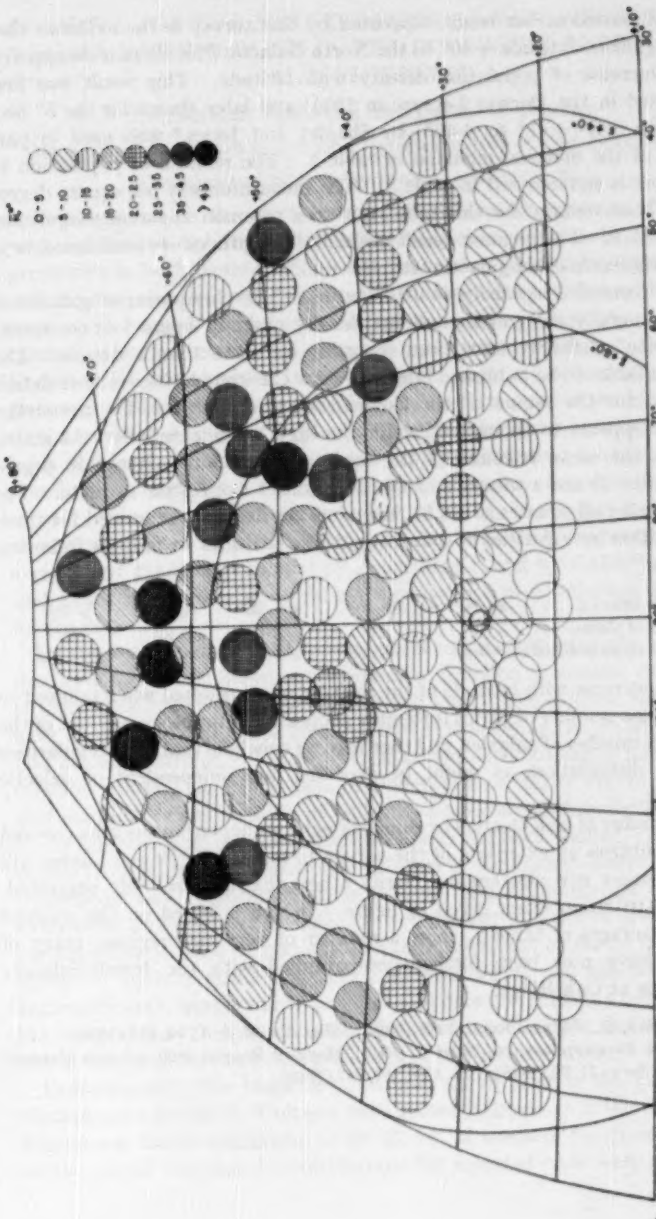


FIGURE 1

Distribution of northern galaxies.

The second earlier result supported by this survey is the evidence that from galactic latitude +40° to the North Galactic Pole there is no appreciable increase of population density with latitude. This result was first reported in the Darwin Lecture in 1934[1] and later shown for the 5° declination belt (+41° to +46°) by Shapley and Jones,[2] who used in part some of the material included in table 1. The relation of population to latitude is summarized in table 3. The mean numbers per square degree have been reduced for this tabulation to a common apparent magnitude, $m_z = 17.9$. The greatest density, when all longitudes are considered, is in the neighborhood of galactic latitude +60°.

6. *General Irregularity in Distribution.*—The clumpiness of galaxies is now generally recognized, and the density chart in figure 1 is consistent with the results we have been reporting for more than a decade. The census tables to be published later will show the irregularities in finer detail. Except for the region of the Cepheus flare, obscuration by interstellar clouds appears to be ineffective for the high latitude regions. As stated above, the mean residuals of the mean numbers for each square degree (see table 2) are a measure of the inequalities from plate to plate. The average for all squares is ±0.51, but when the fields are examined for these population irregularities at different galactic latitudes we find the following results:

Latitude interval.....	20–30°	30–40°	40–50°	50–59°	> 59°
Number of plates.....	42	36	31	22	22
Average mean residual	±0.65	±0.99	±1.16	±1.87	±1.84

The increase with latitude of the average mean residual when reduced to a common number of plates is roughly proportional to the square root of the average number of galaxies, and therefore we conclude that the irregularities in the distribution, as might be expected, are independent of galactic latitude.

A number of rich clusters or clouds of galaxies appear in the area covered and doubtless an extension of the survey to magnitude 19 and fainter will reveal many new clusters and clarify others that are here only suggested. The distribution chart given in figure 1, which is based on the reduced plate averages of table 1, show a number of very rich regions, many of which have now been further photographed with the Jewett-Schmidt telescope at Oak Ridge.

[1] Shapley, H., Harvard Reprint 105 (1934); *Mon. Not. R. A. S.*, **94**, 813 (1934).

[2] These PROCEEDINGS, **26**, 599–604 (1940), Harvard Reprint 209; see also Harvard Reprint, Series II, 23 (1948); *Sci. Mon.*, **67**, 247 (1948).

EVIDENCE FOR THE PARTICIPATION OF KYNURENINE AS A NORMAL INTERMEDIATE IN THE BIOSYNTHESIS OF NIACIN IN NEUROSPORA*

BY CHARLES YANOFSKY AND DAVID M. BONNER

OSBORN BOTANICAL LABORATORY, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT

Communicated by P. R. Burkholder, January 28, 1950

Investigations on the mechanism of the biological synthesis of niacin have implicated tryptophan and 3-hydroxy-anthranilic acid as niacin precursors in both Neurospora[1, 2] and the rat.[3, 4] 3-Hydroxy-anthranilic acid has been shown to serve as a normal intermediate in niacin synthesis in Neurospora,[5] and a mechanism for the conversion of this compound to niacin has been suggested.[6] It was not until recently, however, that conclusive proof was obtained that tryptophan normally serves as a major precursor of niacin in Neurospora.[7] As yet no clear proof of the identity of the intermediates between tryptophan and 3-hydroxy-anthranilic acid has been obtained. Kynurenine and 3-hydroxy-kynurenine have been proposed as intermediates in the conversion of tryptophan to niacin in Neurospora.[8] Kynurenine has been tested for its ability to support the growth of rats maintained on a niacin deficient diet and found inactive.[8] However, recent evidence presented by Heidelberger[9] and by Kallio[10] suggests that kynurenine may play a role in the conversion of tryptophan to niacin by the rat.

In the Neurospora investigations evidence substantiating the proposal that kynurenine and 3-hydroxy-kynurenine function as natural precursors of niacin has not yet been presented. Although both of these substances have been tested and found highly active in supporting the growth of certain niacinless strains,[1, 11] proof of their actual participation in niacin synthesis is still lacking. This lack of evidence is primarily due to the fact that mutant strains capable of utilizing 3-hydroxy-anthranilic acid and niacin but incapable of using tryptophan have not been studied. Mutants of this class might accumulate one or more of the intermediates between tryptophan and 3-hydroxy-anthranilic acid, and accumulation of either kynurenine or 3-hydroxy-kynurenine would constitute direct proof of the participation of these compounds in the biosynthesis of niacin in Neurospora. A mutant strain has been obtained[6] which can use 3-hydroxy-anthranilic acid, cannot use tryptophan and does accumulate a kynurenine-like compound. The present paper deals with the isolation and identification of this substance and the significance of this accumulation.

Experimental.—The basal medium used throughout is the customary Neurospora minimal.[12] Cultures were grown 72 hours at 25°C. in 125-ml. Erlenmeyer flasks containing 20 or 40 ml. of medium for growth tests. At the end of the period of incubation the mycelial pads were removed,

dried and weighed. Test samples were added to the assay flasks before autoclaving.

Table 1 lists the various Neurospora mutants used in this investigation and the substances known to support their growth.

Strain Y-31881 is biochemically distinct from the other mutant strains listed, since the group of compounds which will support its growth is characteristic of it alone. The growth of this strain is supported by any one of the following compounds: 3-hydroxy-kynurenine,† 3-hydroxy-anthranilic acid, quinolinic acid at high concentrations and niacin. Tryptophan, its precursors or kynurenine cannot replace niacin for this mutant. Genetic tests demonstrate that Y-31881 differs from the parental strain by a single gene mutation. That this strain differs genetically from the other strains listed in table 1 was concluded from the fact that it forms a heterocaryon (as a test for allelism)[14] with these strains, and that from crosses with each of the strains listed in table 1, nutritionally wild-type progeny were recovered.

TABLE 1

GROWTH OF VARIOUS STRAINS OF NEUROSPORA

STRAIN NO.	NO ADDITION	ANTHRANILIC ACID	INDOLE	TRYPTOPHAN	KYNURENINE	α-N-ACETYL KYNURENINE	3-HYDROXY-KYNURENINE	3-HYDROXY ANTHRANILIC ACID	QUINOLINIC ACID	NIACIN
75001	−	+	+	+	+	−	?	−	−	−
10575	−	−	+	+	−	−	−	−	−	−
39401	−	−	+	+	+	+	+	+	±	+
Y-31881	−	−	−	−	−	−	+	+	±	+
4540	−	−	−	−	−	−	−	−	±	+
3416	−	−	−	−	−	−	−	−	−	+
Y-31881-3416	−	−	−	−	−	−	−	−	−	+
Wild type	+	+	+	+	+	+	+	+	+	+

On the basis of these observations it was concluded that this mutant is genetically blocked in the conversion of kynurenine to 3-hydroxy-anthranilic acid. The double mutant Y-31881–3416 permitted a direct check on this interpretation. Strain 3416 is a niacin-requiring mutant which cannot use 3-hydroxyanthranilic acid in place of niacin, and accumulates quinolinic acid freely in its culture medium.[6] If the Y-31881 block interfered with the synthesis of 3-hydroxy-anthranilic acid, which in turn is known to give rise to quinolinic acid, it would be expected that this double mutant should no longer accumulate this acid. The opposite mating types of the two strains concerned were crossed and ascospores were isolated in order from several asci. In one ascus four wild-type spores were obtained, indicating that the other four must be double mutants. Crossing a presumed double mutant with wild type gave both individual mutant types, verifying the genotype of the double mutant. Y-31881–3416 and 3416,

one of the parental types, were grown on a medium supplemented with niacin and the amount of quinolinic acid accumulated determined. It can be seen from table 2 that the double mutant Y-31881–3416 does not accumulate an appreciable amount of quinolinic acid when compared to strain 3416. The Y-31881 block must then prevent an enzymatic reaction which normally leads to the production of a quinolinic acid precursor.

Cross-feeding experiments were performed to test for the accumulation of an active niacin precursor in the culture filtrates of strain Y-31881. Strain 39401 was selected as the assay organism since this strain would be expected to grow in the presence of indole, tryptophan or any of the intermediates between tryptophan and niacin (see table 1). The results of a typical experiment are shown in figure 1. It is clear that these filtrates contain a substance or substances which support the growth of strain 39401 but not strain Y-31881. Since the growth of strain 39401 alone is supported by Y-31881 culture filtrate, it may be concluded that the accumulated substance is not 3-hydroxy-kynurenine, 3-hydroxy-anthranilic acid, quinolinic acid or niacin, since both strains are equally sensitive to these compounds.

To obtain sufficient material for the isolation of the active substance (designated as 31881-I), strain Y-31881 was grown in 5-gallon bottles con-

TABLE 2

QUINOLINIC ACID ACCUMULATION BY VARIOUS MUTANT STRAINS OF NEUROSPORA

STRAIN	SUPPLEMENT	γ-QUINOLINIC ACID PER CC. FILTRATE	γ-QUINOLINIC ACID PER MG. DRY WEIGHT
3416	20 γ-nicotinamide	50.5	8.3
Y-31881-3416	20 γ-nicotinamide	0.49	0.14

taining 18 liters of half-strength minimal medium supplemented with 4 mg. of nicotinamide. Half-strength minimal was used instead of normal strength since it did not decrease the amount of 31881-I accumulated yet reduced the difficulty of isolation. The cultures were incubated at 25°C. under continuous aeration. After 7–10 days of growth the mycelium was removed by filtering through cheesecloth. The filtrates from two bottles were combined and concentrated *in vacuo* to approximately two liters. The concentrate was filtered to remove insoluble material, which was then washed twice with ethanol. The washings were added to the clear concentrate. Two volumes of ethanol were added and the mixture cooled overnight. The precipitate formed was filtered off and washed as before. The clear solution was evaporated almost to dryness, brought up to a volume of 300 cc. with distilled water and acidified to pH 3.5–4.0. It was then continuously extracted with ether for 48 hours. The ether extract was concentrated to ca. 10 cc. and enough ethanol added to bring the final volume to 60 cc. This extract was chromatographed using whole sheets of Whatman No. 1 filter paper cut to 17 × 20 inches. Twelve sheets were

run at the same time and constituted a single batch. Five cc. of the ethanol extract were applied to each sheet by means of a modified kymograph previously described.[14] The sheets were then developed as ascending chromatograms with a butyl alcohol, propyl alcohol, water (1:2:1) solvent made 0.005 *M* with respect to ammonia immediately before use. After 24 hours the sheets were removed and air-dried.

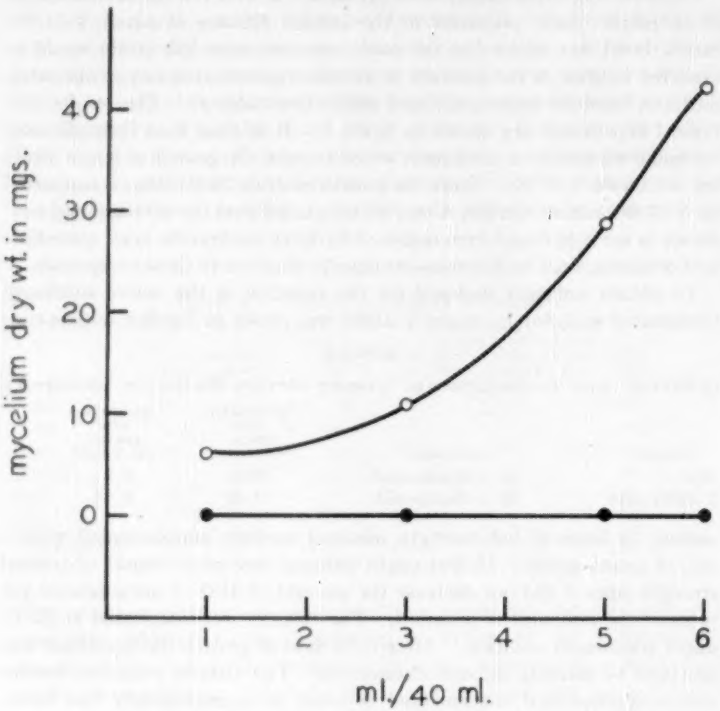


FIGURE 1

Growth of strains 39401 and Y-31881 on culture filtrates of strain Y-31881. ○–○ strain 39401, ●–● strain Y-31881.

The band of active material was next located in the following manner. Sterile modified Neurospora minimal medium was inoculated with a filtered conidial suspension of the test strain and poured into large ($5^1/_2 \times$ 16 inch) plates.[15] The medium used contained inorganic salts, 2% agar, 0.25% sucrose and 1% sorbose, this latter sugar being used to prevent spreading growth of the mycelium.[16] Strains 39401 and Y-31881 were

used as test organisms. Two vertical sections each 1 cm. in width were cut from one sheet of each batch. These strips were sterilized by exposure to a Sterilamp for ca. 15 minutes, after which one strip was placed on a 39401 plate and its duplicate on a Y-31881 plate. Each plate could accommodate 3 strips with sufficient space between them to prevent overlapping of the growth zones. The plates were then maintained at 25°C.

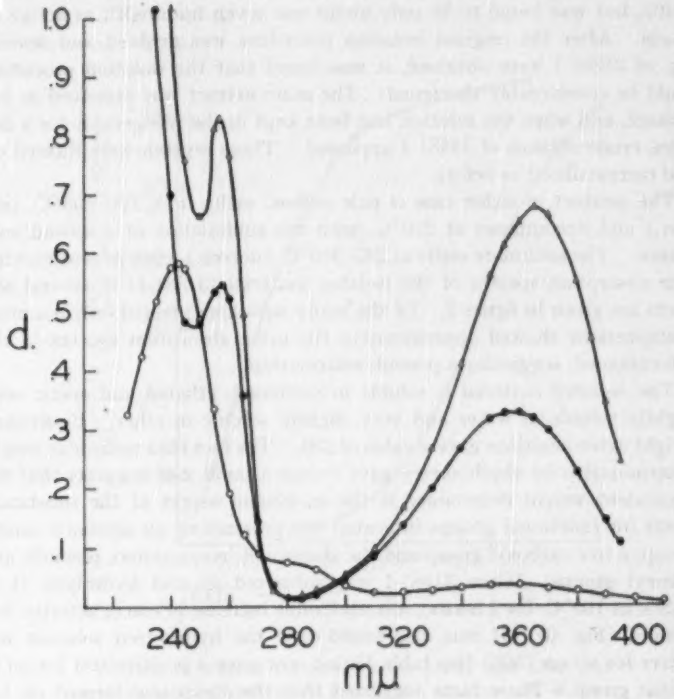


FIGURE 2

Absorption spectra of 31881-I in various solvents. ○–○ 0.1 *N* HCl; ●–● 0.1 *N* NaOH; △–△ abs. ethanol.

for 24 hours. Contaminants were seldom encountered because of the short period of incubation. The areas supporting the growth of 39401 but not Y-31881 were marked and recorded, and the corresponding sections from all sheets of the same batch were cut out and combined. These sections were eluted with dilute ammonia in a Waring blendor. The suspension was filtered to remove the paper pulp, this process repeated

and the eluates combined. The paper pulp was tested for residual activity before being discarded, and it was seldom found necessary to elute a third time. The combined eluates were evaporated to dryness *in vacuo* and the minimum amount of hot ethanol added to effect complete solution. On cooling, light brown needles which supported the growth of strain 39401 appeared. The needles were filtered off and recrystallized from water after norite treatment. This material remained active for growth of strain 39401, but was found to be only about one seven hundredth as active as niacin. After the original isolation procedure was evolved and several mg. of 31881-I were obtained, it was found that the isolation procedure could be considerably shortened. The ether extract was dissolved in hot ethanol, and when the solution had been kept in the refrigerator for a few days, crude crystals of 31881-I appeared. These crystals were filtered off and recrystallized as before.

The product in either case is pale yellow, melts from 190–195°C. (uncorr.) and decomposes at 210°C. with the sublimation of a second substance. The sublimate melts at 237–240°C. (uncorr.) when heated rapidly. The absorption spectra of the isolated material (31881-I) in several solvents are given in figure 2. Of the many substances tested only *o*-aminoacetophenone showed approximately the same absorption spectra in the solvents used, suggesting a possible relationship.

The isolated material is soluble in methanol, ethanol and acetic acid, slightly soluble in water and very slightly soluble in ether. Equivalent weight determinations gave a value of 250. The fact that molecular weight determination by ebulliometry gave values around 250, suggests that the equivalent weight determined is the molecular weight of the substance. Tests for functional groups indicated the presence of an aromatic amino group, a free carboxyl group and the absence of free α-amino, phenolic and alkoxyl groups. When 31881-I was subjected to acid hydrolysis (1 *N* H_2SO_4 at 100°C. for 2 hours), a considerable increase in niacin activity was noticed (Fig. 3). It was also found that the hydrolyzed solution was active for strain 75001 (see table 1) and now gave a positive test for an α-amino group. These facts suggested that the compound formed on hydrolysis was kynurenine. To establish this fact the isolation of kynurenine was attempted. One hundred mg. of 31881-I were hydrolyzed with 1 *N* H_2SO_4, the H_2SO_4 content brought up to 5% (by volume) and enough ethanol added to make an 80% solution. After two days in the refrigerator the colorless crystals were filtered off. These needles gave a strong qualitative test for kynurenine, contained sulfate, were active for 75001 and 39401 and appeared in all tests to be identical with kynurenine sulfate. It was also necessary to determine the nature of the group removed by acid hydrolysis in order to establish the structure of 31881-I. Since the α-amino group was freed by acid hydrolysis it was suspected that the mask-

ing group was either a formyl or an acetyl group. On acid hydrolysis these groups would give rise to formic and acetic acids. A 100-mg. sample of 31881-I (in 10 cc. of 1 *N* H_2SO_4) was hydrolyzed in a sealed glass tube placed in a boiling water bath for $1^1/_2$ hours. The tube was removed, cooled, opened and the contents poured into a distilling flask. The volatile acids were distilled over and the Duclaux distillation constants determined. The constants agreed perfectly with those found using a known

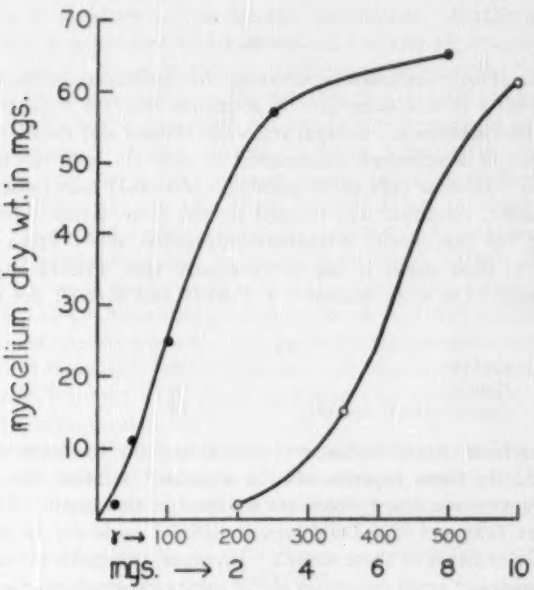


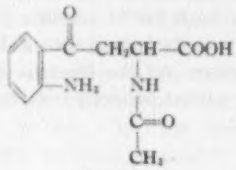
FIGURE 3

Activity of 31881-I for strain 39401 before and after acid hydrolysis. ○–○ before acid hydrolysis—mg. scale; ●–● after acid hydrolysis—γ scale.

acetic acid solution. Furthermore, 0.36 meq. of acid was present in the 100 cc. of distillate. This amount of acetic acid accounts for 90% of the theoretical amount of acid which would be liberated by the complete hydrolysis of 100 mg. of α-N-acetyl kynurenine. C-H analysis of 31881-I is compared below with that calculated for α-N-acetyl kynurenine.

	C	H
Found	57.69	5.77
Calculated for $C_{12}H_{14}O_4N_2$	57.58	5.63

On the basis of these data we have identified 31881-I as α-N-acetyl kynurenine (formula I). Final proof awaits synthesis of this compound, which is now in progress.



Formula I

To arrive at any conclusions concerning the significance of the accumulation of 31881-I it was necessary to determine whether other strains accumulate this substance. Several years ago Bonner and Beadle[17] reported the isolation of a substance (designated as 4540-II) from the filtrates of strain 4540. Because only small quantities of 4540-II were present and its activity slight, attention was focused on the more active substance accumulated by this strain, 3-hydroxy-anthranilic acid. From the data presented in their paper it can be concluded that 4540-II and 31881-I are identical. The C-H analyses of 4540-II and 31881-I are compared below:

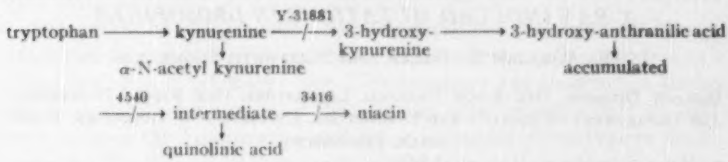
	C	H
4540-II	57.55	5.68
31881-I	57.69	5.77
Calculated for $C_{12}H_{14}O_4N_2$	57.58	5.63

Filtrates from other strains were also tested for the presence of this compound. In these experiments the standard isolation and detection procedures were employed which are reported in this paper. The strains tested were 2198 and 5256 (wild type). 31881-I could not be detected in the filtrates of either of these strains. However, this method could not be expected to detect small quantities of this substance which may be normally formed.

Discussion.—Tryptophan and kynurenine have been proposed as intermediates in the synthesis of niacin in Neurospora largely because they support the growth of one genetic type of niacin-requiring mutant strain.[1] 3-Hydroxy-anthranilic acid similarly shows niacin activity[2] but proof of its natural participation in niacin synthesis has been obtained.[3] On the basis of these observations a scheme was proposed involving tryptophan, kynurenine, 3-hydroxy-kynurenine and 3-hydroxy-anthranilic acid as precursors of niacin. Certain observations regarding the strain upon which the participation of tryptophan and kynurenine have been predicated are not readily reconcilable with the thesis that tryptophan and kynurenine serve as major niacin precursors.[5,18] However, N^{15} experiments, designed specifically to test this point, have shown conclusively

that tryptophan is the main source of niacin in Neurospora.[7] Also consistent with the proposed scheme is the fact that 3-hydroxy-kynurenine has been found active in replacing niacin.[11] However, final proof for the participation of kynurenine and 3-hydroxy-kynurenine depends upon the demonstration that these compounds can be and are synthesized by Neurospora. The work reported in this paper has shown that a metabolite of kynurenine is abnormally accumulated as a result of a genetic block in one strain. Therefore, proof has been obtained for the ability of the mold Neurospora to produce kynurenine-like compounds. Furthermore, this accumulation is associated with the inability of the strain concerned to synthesize niacin. There is a direct relationship between the two. Consistent with these observations is the fact that this mutant will grow in the presence of 3-hydroxy-kynurenine but not in the presence of kynurenine. Hence, it apparently cannot convert the latter substance to the former. Thus, it has been shown that Neurospora can and does synthesize kynurenine-like compounds and that these compounds are involved in niacin synthesis. In rats, the normal excretory product of kynurenine is either kynurenic or xanthurenic acid, depending upon the B_6 content of the diet. In view of the small quantities of α-N-acetyl kynurenine formed by strain 4540,and the larger quantities formed by strain Y-31881, it appears that this compound represents a major end-product of kynurenine metabolism in Neurospora. However, both kynurenic and xanthurenic acids might reasonably be expected to be found as kynurenine metabolites in Neurospora under certain conditions.

The following scheme represents the present status of the investigations on niacin synthesis in Neurospora. The vertical arrows indicate compounds accumulated as a result of the indicated genetic block.



Summary.—The isolation and identification of a substance possessing slight niacin activity for one mutant strain of Neurospora has been described. The strain accumulating this substance can synthesize niacin from 3-hydroxy-kynurenine or 3-hydroxy-anthranilic acid but is incapable of utilizing tryptophan or kynurenine for this purpose. If the proposed pathway from tryptophan to niacin in Neurospora is correct the accumulated compound should be kynurenine or some product of kynurenine metabolism. Such a product has been isolated and identified as α-N-acetyl kynurenine. The accumulation of this kynurenine-like compound as a result of an induced genetic block is considered direct evidence for the participation of kynurenine in the biosynthesis of niacin in Neurospora.

* These investigations were supported in part by the Williams-Waterman Fund for the combat of dietary diseases, and in part by the Rockefeller Foundation.

† The authors wish to thank Dr. W. Weidel of the Kaiser-Wilhelm Institut für Biochemie, who generously made a sample of 3-hydroxykynurenine available to us.

[1] Beadle, G. W., Mitchell, H. K., and Nyc, J. F., PROC. NATL. ACAD. SCI., **33**, 155 (1947).

[2] Mitchell, H. K., and Nyc, J. F., *Ibid.*, **34**, 1 (1948).

[3] Krehl, W. A., Tepley, L. J., Sarma, P. S., and Elvehjem, C. A., *Science*, **101**, 489 (1945).

[4] Mitchell, H. K., Nyc, J. F., and Owen, R. D., *J. Biol. Chem.*, **175**, 433 (1948).

[5] Bonner, D., PROC. NATL. ACAD. SCI., **34**, 5 (1948).

[6] Bonner, D. M., and Yanofsky, C., *Ibid.*, **35**, 576 (1949).

[7] Bonner, D. M., and Partridge, C. W. H., unpublished.

[8] Krehl, W. A., Bonner, D. M., and Yanofsky, C., *J. Nutrition*, in press.

[9] Heidelberger, C., Gullberg, M. E., Morgan, A. F., and Lepkovsky, S., *J. Biol. Chem.*, **179**, 143 (1949); Heidelberger, C., Abraham, E. P., and Lepkovsky, S., *Ibid.*, **179**, 151 (1949).

[10] Kallio, R. E., and Berg, C. P., *Ibid.*, **181**, 333 (1949).

[11] Haskins, F. A., and Mitchell, H. K., PROC. NATL. ACAD. SCI., **35**, 500 (1949).

[12] Beadle, G. W., and Tatum, E. L., *Am. J. Botany*, **32**, 678 (1945).

[13] Beadle, G. W., and Coonradt, V. L., *Genetics*, **29**, 291 (1944).

[14] Yanofsky, C., Wasserman, E., and Bonner, D. M., *Science*, **111**, 61 (1950).

[15] Beadle, G. W., Mitchell, H. K., and Bonner, D. M., *J. Bact.*, **49**, 101 (1945).

[16] Tatum, E. L., Barratt, R. W., and Cutter, V. M., Jr., *Science*, **109**, 509 (1949).

[17] Bonner, D., and Beadle, G. W., *Arch. Biochem.*, **11**, 319 (1946).

[18] Bonner, D. M., and Wasserman, E., *J. Biol. Chem.*, in press.

THE EFFECT OF OXYGEN CONCENTRATION ON THE RATE OF X-RAY INDUCED MUTATIONS IN DROSOPHILA

BY WILLIAM K. BAKER AND ELIZABETH SGOURAKIS

BIOLOGY DIVISION, OAK RIDGE NATIONAL LABORATORY, OAK RIDGE, TENNESSEE,* AND DEPARTMENT OF ZOOLOGY AND ENTOMOLOGY, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE†

Communicated by J. T. Patterson, January 10, 1950

Various supplemental agents have been administered to organisms being exposed to x-rays and the effects of these agents on the induced frequency of gene mutations and chromosome aberrations have been studied. In general, the supplemental treatments which did alter the x-ray-induced mutation or chromosome rearrangement rate were found to increase this rate. However, the recent work of Thoday and Read,[1] Hayden and Smith,[2] and Giles and Riley[3] provide evidence that by lowering the oxygen tension during x-ray exposure of plant material, the induced frequency of

chromosome aberrations is greatly reduced. In view of the implications of this finding, it seemed advisable to determine if a lowered oxygen concentration reduces the radiation damage to the genes in animal material.

Experimental Methods.—The induction of sex-linked, recessive, lethal mutations was investigated in the Oregon-*R* strain of *Drosophila melanogaster*. Wild-type males of this strain (from one to three days in age) were treated with x-rays in the manner described below and then immediately mated in a culture bottle to virgin females of the Muller-5 stock ($sc^{sl}B$ *In-S* w^a sc^s)[4]. Seven to eight days after the time of treatment the parents were removed from the culture bottle. The F_1 females, all of which carry an irradiated *X*-chromosome, were aged with their brothers (Muller-5 males) for at least two days in a fresh culture bottle in order to increase the chance of insemination. Approximately the first 200 F_1 females which hatched from the parent culture were separated individually after aging into shell vials containing one or two Muller-5 males. A preliminary examination was made of the F_2 offspring for the presence of wild-type males by viewing the flies inside the culture vial with a binocular dissecting microscope. If one or more wild-type males were observed, the *X*-chromosome being tested was classified as a chromosome with no lethal mutations. On the other hand, if no wild-type males were observed in the preliminary examination of the F_2 flies, the culture was set aside until all the offspring had hatched and then the flies were anesthetized and classified. If this examination verified the absence of wild-type males, a further test of this lethal-bearing *X*-chromosome was made by mating in a shell vial three of the F_2 females which were heterozygous for the Muller-5 chromosome to five of their Muller-5 brothers. The F_2 offspring were counted to substantiate the presence of a lethal-bearing chromosome.

Because of the intergradation between mutations that have a full lethal effect and those which are semilethal, the classification of a mutation as a lethal mutation is arbitrary. An *X*-chromosome was classified as having one or more lethal mutations if the following criterion was valid: no wild-type males in the F_2 generation and 5% or less males of this type in the F_3. Less than 2% of the F_2 cultures with no wild-type males contained less than twelve Muller-5 males. These were classified as lethals only if 15 or more offspring were present; otherwise, they were classed as failures.

X-radiation was administered to the flies by means of a General Electric, Maximar Model, 250-kvp. unit which contained a self-rectifying Coolidge tube with a tungsten target. The inherent filtration was equivalent to 3 mm. of aluminum. In all the experiments 250-kvp. x-rays, produced by a 15-ma current, were used. This gave a dosage rate of about 125 r per minute at the target distance of 61.5 cm. The dosage rate was kept constant in all experiments.

During irradiation one group of flies was maintained in a continuous flow of oxygen gas and another group in a continuous flow of nitrogen gas; both groups being kept at a known constant temperature by means of the apparatus to be described. The gases, obtained from commercial cylinders, were passed through copper coils which were immersed in a water-bath containing either tap water or a mixture of tap water and ice. A beaker with two pairs of small side arms was also partially immersed in this water-bath. The gas, after passing through the copper coils, entered the beaker by means of one of the side arms. Between this side arm and the one on the opposite side of the beaker, was stretched a rubber balloon (open at both ends) which contained the flies enclosed in a gauze capsule. After the gas passed over the flies, it was conducted through copper tubing from the beaker to the outside of the water-bath container. Two sets of copper coils and four side arms on the beaker made possible the simultaneous exposure to x-rays of one group of flies in oxygen and another group in nitrogen, both groups being maintained at the same temperature. The top of the beaker, which extended just above the surface of the water-bath, was covered by a lucite lid and the radiation was administered to the flies through this window. In addition to the four small side arms previously mentioned, there was one large side arm on the beaker which extended from the beaker to the outside of the water-bath. This side arm allowed insertion of a thimble chamber into the beaker at the location where the flies received the radiation and also made possible the insertion of a thermometer into the beaker for temperature measurements.

The treatment procedure may be summarized as follows. The irradiation apparatus was placed underneath the target port and the dosage being delivered was determined by placing the thimble chamber into the beaker. The thimble chamber and the electrometer used had recently been standardized. The calculated dosage to be given was obtained by averaging the five dosimeter readings taken. About 150 Oregon-*R* males were placed in each of two gauze capsules and each capsule inserted into one of the balloons. The gases were then passed across the flies and ten minutes later the temperature inside the beaker was checked and the x-ray exposure begun. After the time necessary for the desired dosage had elapsed, the males were removed from the gas and mated immediately to Muller-5 females.

Experimental Results.—The data were gathered by means of 31 individual experiments of the type described above. Chi-square tests of homogeneity of the three or more experiments conducted with a given gas, temperature and dosage show no indication of heterogeneity among the individual experiments. Therefore, it is permissible to lump the results from the individual experiments in which the flies received the same treatment. The lumped data showing the effects of oxygen concentration, temperature

and dosage on the frequency of induced lethal mutations are compiled in table 1. The striking reduction in the number of lethal mutations induced when the flies are maintained in a nearly oxygen-free atmosphere is clearly evident from this table and the curves presented in figure 1. Chi-square values obtained by testing the independence of the gas used on the mutation frequency are tabulated in the seventh column of this table. The reduction in the mutation rate in the nitrogen series is highly significant ($P < 0.01$) except in the experiments in which the flies were given 1000 r

TABLE 1

EFFECT OF O_2 CONCENTRATION, TEMPERATURE AND DOSAGE ON LETHAL MUTATION RATE

DOSAGE	TEMPERATURE RANGE	GAS	NO. CHROMOSOMES TESTED	NUMBER LETHALS	PER CENT LETHALS	χ^2 GAS	χ^2 TEMP.
Control	0.5– 1.0	O_2	602	1	0.2		...
						...	
	0.5– 1.0	N_2	608	2	0.3		
	26.0–28.5	O_2	607	1	0.2		...
						...	
	26.0–28.5	N_2	605	1	0.2		
1000 r	1.5– 3.0	O_2	600	33	5.5		O_2 1.6
						7.4	
	1.5– 3.0	N_2	582	14	2.4		
	22.0–28.0	O_2	605	24	4.0		N_2 0.0
						2.5	
	22.0–28.0	N_2	594	14	2.4		
3000 r	1.0– 3.0	O_2	485	88	18.1		O_2 7.8
						36.4	
	1.0– 3.0	N_2	767	54	7.0		
	26.0–28.2	O_2	603	73	12.1		N_2 0.1
						9.0	
	26.0–28.2	N_2	775	56	7.2		
5000 r	1.0– 3.0	O_2	105	24	22.9		O_2 1.2
						40.3	
	1.0– 3.0	N_2	1206	119	9.9		
	23.0–29.0	O_2	429	78	18.2		N_2 3.8
						16.8	
	23.0–29.0	N_2	1523	118	7.7		

units and kept at the warm temperature. There seems no reason to doubt that more extensive experiments conducted at this dosage and temperature would provide a statistically significant difference. The results assembled in table 1 also furnish direct evidence, which substantiates the claim of some previous investigators,[5] that more mutations are induced when the flies are kept at a near freezing temperature than are induced at room temperature. The chi-square values obtained from tests of independence of temperature on mutation rate are tabulated in the last column of table 1.

At all three dosage levels more mutations are produced at the lower temperature when the flies are irradiated in an oxygen atmosphere, but the difference is statistically highly significant only in the 3000 r series. In the nitrogen-treated flies, on which more extensive data are available at the higher dosages, the temperature difference causes no apparent alteration in the mutation rate. At 5000 r units the chi-square value is on the borderline of the conventional 5% level of significance. These results seem to indicate that the "temperature effect" is actually an effect of altered oxygen tension in the sperm possibly caused by differences in oxygen solubility

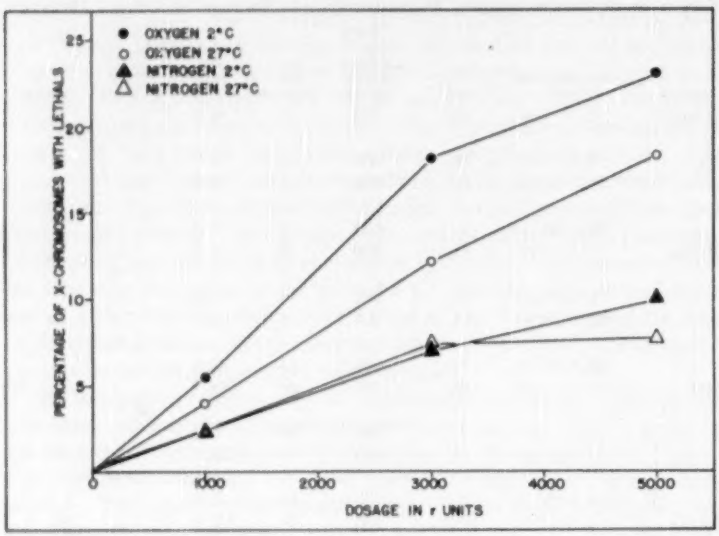


FIGURE 1

X-ray dosage curves for the lethal mutations induced in *Drosophila* irradiated in oxygen or nitrogen gas at 2° and 27° C.

and rate of respiration at the two temperatures. It should be noted that the control experiments conducted without irradiation are not sufficiently large to give any reliable information concerning a possible relation between oxygen tension and spontaneous mutation rate.

It is evident from the figures presented in table 1 that at the higher dosages fewer chromosomes were tested in the oxygen-treated flies than in the nitrogen series. Although a large number of experiments were conducted with the flies in oxygen, only a relatively small number of F_1 offspring were produced from the males which had been maintained in this

gas during irradiation. The reduced number of offspring in the oxygen as compared to the nitrogen series could possibly be due to any of four factors: (1) reluctance of the males to mate, (2) sperm immotility,[6] (3) lowered viability of the treated males, or (4) increase in the frequency of dominant mutations and chromosome aberrations which are lethal. Experiments were undertaken to determine which of these factors was effective. After irradiation of the Oregon-*R* males with 5000 r units at 27°C., sixty matings

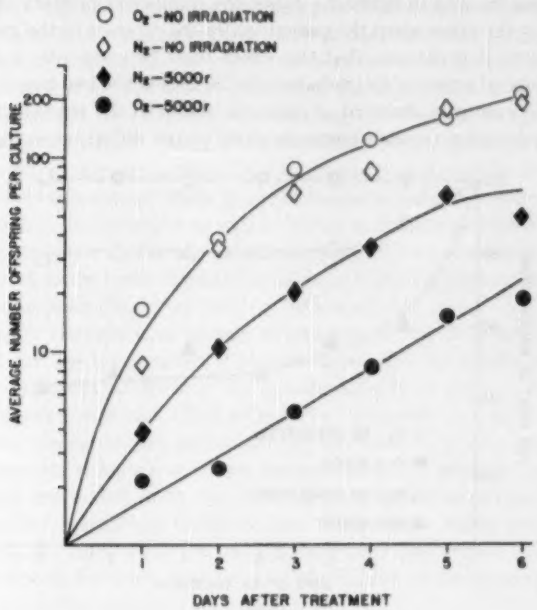


FIGURE 2

Relationship between the oxygen concentration at time of irradiation and the number of offspring produced. On the abscissa is plotted the number of days after treatment during which the females were allowed to deposit eggs.

were made; each mating consisting of three males treated in oxygen and five Muller-5 females placed together in a shell vial. Another 60 similar matings were made using the males which had been exposed in an atmosphere of nitrogen. Immediately after treatment the males in both series were checked to see if any had been killed during irradiation. Every 24 hours after exposure, the number of males living in each culture was observed; also, ten cultures were selected at random in each gas series and the reproductive tracts of the five females in each culture were dissected out

and examined for the presence and motility of sperm. This procedure was repeated every 24 hours after treatment for six days. The number of offspring produced by each culture was counted. This gives an estimate of the number of offspring per culture produced from the eggs laid during the first 24, 48, 72, etc., hours after treatment.

The smaller number of offspring per culture produced by males which were exposed to 5000 r units of x-rays while in oxygen as compared with nitrogen can be seen in figure 2. Since the number of progeny in the two gas series is the same when the parent males are exposed to the gases without radiation, it is obvious that this effect must be caused by a difference in the biological action of the radiation on the flies in the two gases.

Only one case was observed of immotile sperm in the reproductive tract of the female and no evidence was obtained of any difference in the number

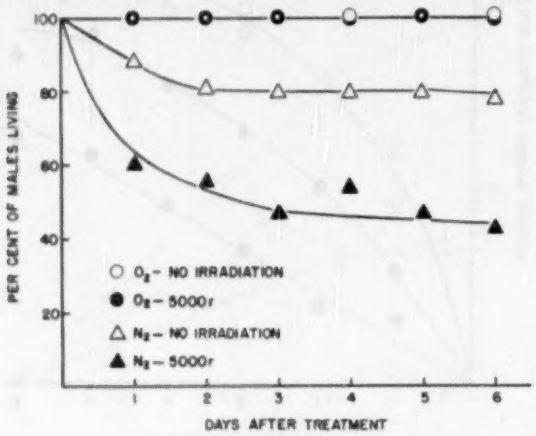


FIGURE 3

Relationship between the oxygen concentration at time of irradiation and the viability of the treated males.

of females inseminated; between 90 and 100% of the females were inseminated in both series from the first day on. These observations eliminate the possibility that the first two factors listed above are effective. The data presented in figure 3 indicate that male viability is not an important factor since it acts in the opposite direction. About one-half of the males exposed in nitrogen die during the first few days, while the viability of the flies in oxygen is not affected over the course of the experiment. Incidentally, it appears from a comparison of the two nitrogen curves in figure 3 that the differential killing is not only caused by the gas alone but is enhanced by the radiation. These findings then provide evi-

dence that the difference in the number of offspring produced from males treated in the two gases must be attributed to a reduction in the frequency of induced dominant lethal mutations when the flies are irradiated in nitrogen. Further evidence, based on the smaller number of failures of the F_1 matings in the nitrogen series, also indicates a lowering of the frequency of dominant sterility mutations in the absence of oxygen.

Discussion.—It appears reasonable to attribute the reduction in the frequency of mutations induced in the male flies irradiated in a nitrogen atmosphere to a lowered oxygen tension in their sperm. It must be assumed then that the nitrogen acts as an inert gas in so far as this effect is concerned and, also, the assumption must be made that, by greatly reducing the oxygen concentration surrounding the flies, the oxygen tension in the sperm is lowered. Since Giles and Riley[3] found that the reduction, as compared to air or oxygen, in the number of induced chromosome aberrations was evident when the *Tradescantia* influorescences were irradiated in helium and argon as well as nitrogen, there is no reason to doubt the first assumption. The second assumption seems most likely since the gas transport to the body tissues in small insects with a tracheal respiratory system takes place almost entirely by diffusion.[7]

The higher frequency of mutations induced in the flies maintained in oxygen at the low temperature is in agreement with the results expected if oxygen is an agent influencing the radiosensitivity of genes. A lowering of the temperature would affect at least two processes in such a way that the oxygen tension in the cell would be increased: (1) the solubility of oxygen in water is higher at colder temperatures, (2) with the low rate of metabolism associated with the cold temperature, the oxygen gradient across the cell membrane would be less than at the warm temperature. This would, in turn, raise the oxygen concentration within the cell. It should be noted, however, that the rate of diffusion of the oxygen would be lower at the colder temperature.

If the oxygen tension within the cell is a factor in determining the sensitivity of genes to x-rays, the question arises as to its mode of action. At the present time any answer to this question is purely speculative. Increased amounts of dissolved oxygen in the cellular fluids being irradiated would form more of the free radicals which act as strong oxidizing agents and which are known to be produced in water by x-radiation. Barron, *et al.*,[8] have published evidence which indicates that it is the oxidizing properties of these radicals which cause the x-ray inhibition of the sulfhydryl enzymes. If such enzymes are necessary in the process of reduplication of particular genes, their inhibition could produce a lethal mutation. On the other hand, recent experiments[9] indicate that organic peroxides, which have rather poor oxidizing properties, are effective mutagenic agents. Thus upon irradiation of cells with high oxygen tensions, the increased amounts

of organic peroxides formed would serve to increase the mutation rate. To date, there is no evidence on which the validity of either mode of action can be established.

Summary.—There is a striking reduction in the number of recessive sex-linked lethal mutations induced in *D. melanogaster* males when they are exposed to x-rays while in an atmosphere of low oxygen concentration. Although an increased number of mutations were induced in flies irradiated in oxygen at 2°C. over those treated in oxygen at 27°C., this increase is not due to the temperature *per se* but rather it is apparently caused by a higher oxygen tension within the irradiated sperm at the lower temperature. Additional evidence also indicates that fewer dominant lethal mutations and chromosome aberrations are induced in flies maintained in a near oxygen-free atmosphere during irradiation.

* This work was done under Contract No. W-7405-eng-26 for the Atomic Energy Commission, Oak Ridge, Tennessee.

† Contribution No. 32 from the Department of Zoology and Entomology.

[1] Thoday, J. M., and Read, J., *Nature*, **160**, 608 (1947).

[2] Hayden, B., and Smith, L., *Genetics*, **34**, 26 (1949).

[3] Giles, N. H., and Riley, H. P., PROC. NATL. ACAD. SCI., **35**, 640 (1949).

[4] For a complete description of the Muller-5 method see Spencer, W. P., and Stern, C., *Genetics*, **33**, 43 (1948).

[5] For a review of this subject see Baker, W. K., *Ibid.*, **34**, 167 (1949).

[6] There is a remote possibility that the oxygen gas might affect the ability of the sperm to fertilize the eggs without disturbing their motility.

[7] Krogh, A., *The Comparative Physiology of Respiratory Mechanisms*, Univ. of Pennsylvania Press (1941).

[8] Barron, E. S. G., Dickman, S., Muntz, J. A., and Singer, T. P., *J. Gen. Physiol.*, **32**, 537 (1949).

[9] Dickey, F. H., Cleland, G. H., and Lotz, C., PROC. NATL. ACAD. SCI., **35**, 581 (1949).

SOME REMARKS ON THE INFINITE DE SITTER SPACE

BY E. P. WIGNER

PALMER PHYSICAL LABORATORY, PRINCETON

Communicated January 20, 1950

One can define de Sitter space as a four-dimensional space with the following two properties. First, it is invariant under the operations of a transitive ten-parametric group. Four of the infinitesimal operators of this group are usually made to correspond to the components of the energy-momentum vector, the other six to the angular momentum tensor. Second, the subgroup of this ten-parametric group which leaves a given point of the space invariant must be isomorphic to the ordinary homogeneous Lorentz

group. This last point ensures that the neighborhood of any point of these de Sitter spaces behaves like the flat space of special relativity (Minkowski space).

The best known examples of de Sitter space in the above sense are, first, the flat space of special relativity, and, second, the ordinary de Sitter spaces. In the former case the ten-parametric group is the inhomogeneous Lorentz group, i.e., the union of Lorentz transformations with the displacements in the four directions of space and time. This space can be regarded as a special case of the ordinary de Sitter spaces. The latter can be represented as the four-dimensional surface

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 = a^2 \qquad (1)$$

in five-dimensional space. The ten-parametric group is, in this case, the $4 + 1$ dimensional homogeneous Lorentz group, i.e., the group of linear homogeneous transformations which leave the form (1) invariant. Given an arbitrary point of this space, e.g., $x_4 = a$, $x_1 = x_2 = x_3 = x_5 = 0$, the subgroup which leaves it invariant is indeed the ordinary homogeneous ($3 + 1$ dimensional) Lorentz group: it is the group which leaves $x_1^2 + x_2^2 + x_3^2 - x_5^2$ invariant. One sees that the ordinary de Sitter space is in a sense more symmetric than the Minkowski space because all the infinitesimal elements of its group are on the same footing. Four of the infinitesimal elements of Minkowski space, i.e., the translations, are distinguished inasmuch as they form a four-parametric commutative subgroup. The Minkowski space can be obtained from the ordinary de Sitter space by setting $a = \infty$ in (1) and restricting the space with the conditions $x_1, x_2, x_3, x_5 \ll a, x_4 \approx a$, i.e., to the neighborhood of a point.

The de Sitter space (1) has a finite extension in the sense that, given any point P (e.g., the point $x_4 = a; x_1 = x_2 = x_3 = x_5 = 0$) and any time-like direction in that point (e.g., the intersection of (1) with the x_4x_5 plane), the geodesics through P, which are perpendicular to the chosen time-like direction, are finite. Three such geodesics in the above case are the intersections of (1) with the x_1x_4, x_2x_4, x_3x_4 planes. These geodesics form circles in the underlying five-dimensional space and return to P after a distance $2\pi a$. Neither is it possible to make this space infinite by considering the point at the distance $2\pi a$ on the geodesic which is, for example, in the x_1x_4 plane, to be different from the original point P. The reason is that one can deform the geodesic in question continuously into a curve of zero length by tilting the plane x_1x_4 about the $x_4 = a$, $x_2 = x_3 = x_5 = 0$ axis in the x_2 direction, i.e., by considering the intersection of (1) with the planes

$$x_3 = x_5 = 0, \qquad x_2 = (x_4 - a)\, tg\vartheta \qquad (2)$$

and increasing ϑ from 0 to $\pi/2$. Hence the ordinary de Sitter space (1) is essentially finite in extension.

The question arises, therefore, whether or not there is a really infinite de Sitter space. The purpose of the present note is to point out that there is. It is formed by the four-dimensional surface

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 = -a^2 \qquad (3)$$

in five-dimensional space. Its group is the 3 + 2 dimensional Lorentz group, i.e., the group of linear homogeneous transformations which leave (3) invariant. Choosing an arbitrary point P on (3), e.g., the point $x_4 = a$, $x_1 = x_2 = x_3 = x_5 = 0$, the subgroup which leaves this point invariant is again the ordinary homogeneous Lorentz group.

The space (3) is contained in the enumerations of both Friedmann[1] and Robertson.[2] It is clearly infinite. If one considers the point P as above, and as time-like direction the intersection of (3) with the x_4x_5 plane, geodesics through P which are perpendicular to the chosen time-like direction are

$$x_4 = a \operatorname{Ch}\varphi; \quad x_\alpha = a \operatorname{Sh}\varphi; \text{ all other } x = 0 \qquad (4)$$

with $\alpha = 1, 2, 3$. These have infinite lengths and are completely space-like (i.e., there is no time-like line through any two of their points).

By suppressing the coordinates x_2 and x_3, one can make a diagram of the surface (3) which looks exactly like the corresponding diagram obtained by setting $x_2 = x_3 = 0$ in (1). However, the geodesics which are space-like if the diagram is considered to represent the finite de Sitter space, are time-like if it is considered to be a representation of the infinite de Sitter space (3). The diagram, figure 1, shows indeed that the space of (3) is infinite but also shows that its time is finite or, rather, periodic.

Geometrically, this is not necessarily true because one can replace the space (5) with its covering space. This can be done, for instance, by introducing new variables

$$x_4 = \rho \cos\vartheta; \qquad x_5 = \rho \sin\vartheta \qquad (5)$$

and not identifying the points the ϑ of which differs by 2π. In contrast to the de Sitter space (1), this is a possible procedure because the different signature of the suppressed coordinates x_2 and x_3 in (3) prevents the line (5) in (3) to be contracted. However, it is doubtful whether in *non quantum* theory the covering surface differs essentially from the original surface because all geodesics (which are the intersections of (3) with planes through the origin $x_1 = x_2 = x_3 = x_4 = x_5 = 0$) which start from a point P will return to that point. As a result, at least in the approximation in which the world lines are geodesics, the world is periodic with the period $2\pi a$ (actually with the period πa) no mater what the initial distribution of the masses is.

The purpose of the present note is to point out that there is an essential difference between the properties of the infinite de Sitter space as viewed from the *quantum* rather than the *non quantum* theoretic point of view. The reason is that not all wave functions have to be periodic in the ϑ of (5) with a period 2π, in fact they can differ by any factor of modulus 1 between two points of the covering space which correspond to the same point P of the space (3). This is equivalent with the statement that the wave functions, if regarded as functions of, say, x_1, x_2, x_3, ρ and ϑ, need not be periodic in ϑ with a 2π period.

There is one further remarkable fact about wave functions in either the space of (3) or in its covering space. The wave functions of "elementary systems" in (3) are given by the irreducible representations of the group of (3), that is the $3 + 2$ dimensional Lorentz group. Bargmann and G. W. Whitehead[3] have determined the topology of this group: It consists of the direct product of a three-dimensional rotation group (2 parameters), of a six-dimensional euclidean space (6 parameters) and of a two-dimensional rotation group (1 parameter). The representations up to a factor of this group have, therefore, one- or two-valued character, corresponding to the presence of the three-dimensional rotation group in the topology of the group of (3). They have, furthermore, one of an infinite number of characters corresponding to the infinitely many-valued representations of the two-dimensional rotation group. These correspond to the possibilities of representing the subgroup of rotations in the x_4x_5 plane by any of the matrices $e^{i\kappa\varphi}$, where κ is not necessarily an integer but can be, in fact, entirely arbitrary. In every irreducible representation of the whole group only those representations of the subgroup will occur in which the fractional part of κ is the same, i.e., which differ by an integer. As a consequence, the fractional part of κ is, in the space (3), an integral of motion, just as the integer or half integer character of the spin is an integral of motion in ordinary quantum theory.

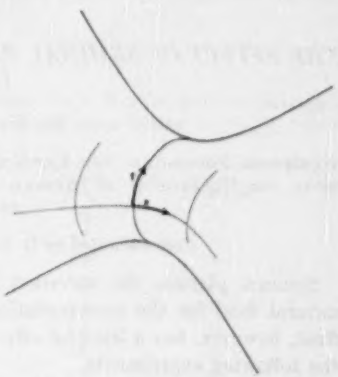


FIGURE 1

If we consider, on the other hand, the physical space as the covering space of (3), the existence of the different types of representations will remove the periodic character of the world. The extra integrals of motion will remain, however, because those transformations of the covering space,

which leave the subspace (3) invariant, commute with all symmetry operations of the group of the covering space.

There is no obvious physical interpretation which one could give these integrals of motion. The fact that the topological structure of the world influences in quantum theory the integrals of motion remains remarkable enough to be recorded.

[1] Friedmann, A., *Z. Physik*, **10**, 377 (1922); **21**, 326 (1924).
[2] Robertson, H. P., *Rev. Mod. Phys.*, **5**, 62 (1933).
[3] Unpublished, personal communication of Dr. V. Bargmann.

*THE EFFECT OF SEMINAL PLASMA ON FERTILIZED RABBIT OVA**

BY M. C. CHANG

WORCESTER FOUNDATION FOR EXPERIMENTAL BIOLOGY, SHREWSBURY, MASSACHUSETTS, AND DEPARTMENT OF PHYSIOLOGY, TUFTS MEDICAL SCHOOL BOSTON, MASSACHUSETTS

Communicated by G. B. Wislocki, January 11, 1950

Seminal plasma, the secretion of the male accessory glands, is the natural fluid for the transportation and survival of spermatozoa. This fluid, however, has a harmful effect on fertilized ova as demonstrated in the following experiments.

Methods.—Fertilized rabbit ova (in 2 cells) were flushed out with rabbit serum from the fallopian tubes of superovulated does about 15 hours after ovulation. The seminal plasma was obtained by centrifugation of fresh human, bull or rabbit whole semen. The seminal plasma of vasectomized rabbits without centrifugation was also used. The seminal plasma in undiluted form or after heat treatment, or diluted with fresh rabbit serum, was used to culture freshly recovered ova in a Carrel flask for 1 day at 38°C. To prevent infection, penicillin was added. Ordinarily, 2-celled ova, cultured in rabbit serum for 1 day, cleave into 12 to 16 cells or occasionally into 32 cells. In the present study, all ova which had divided into 7 cells or more were arbitrarily classified as normal, while those which showed cleavage of less than 7 were considered abnormal.

Results: Ovum Culture in Seminal Plasma.—Table 1 shows the results obtained when fertilized rabbit ova were cultured in seminal plasma of man, rabbit and bull, diluted with rabbit serum. The harmful effect of heterologous as well as homologous seminal plasma on rabbit ova is clearly shown. All of the ova disintegrated in undiluted rabbit or bull seminal plasma, and only 1 out of 16 cleaved normally in human seminal plasma. When bull or rabbit seminal plasma was diluted with 75% rabbit serum,

none of the ova divided normally in bull plasma, whereas half underwent cleavage in rabbit plasma. Thus, the strength or concentration of the harmful substance varies in different species in ascending order as follows: human, rabbit and bull.

The inhibition of cleavage and disintegration of ova in seminal plasma are not due to lack of nutrients, but to the presence of a harmful substance, because ova divided normally in 0.9% NaCl containing 25% rabbit serum. Since seminal plasma of both vasectomized and normal rabbits showed similar harmful effects (table 1), the inhibitory substance must be a component of the secretion of the accessory glands instead of a product of the spermatozoa.

TABLE 1

CLEAVAGE OF FERTILIZED RABBIT OVA CULTURED FOR 1 DAY AT 38°C. IN SEMINAL PLASMA VARIOUSLY DILUTED WITH RABBIT SERUM

SEMINAL PLASMA OF	DILUTED WITH RABBIT SERUM %	TOTAL NO. OF OVA	NO. SHOWING NORMAL CLEAVAGE
Man	0	16	1
	50	17	8
	75	14	9
	87.5	9	7
Rabbit	0	18	0
	50	22	0
	75	19	9
	87.5	10	6
Vasectomized rabbit	0	10	0
	50	9	0
	75	17	11
	87.5	16	12
Bull	0	17	0
	50	22	0
	75	21	0
	87.5	19	6

When rabbit ova were treated with undiluted rabbit seminal plasma for 10, 20, 40 or 80 minutes at 30°C. and then cultured in serum for 1 day, 40 out of 54 (74%) cleaved. It seems that, unlike the ovocidal factor present in the heterologous serum,[1] the harmful substance in the seminal plasma is not immediately lethal, but exerts its harmful effect after a longer period of time.

Characteristics of Ovocidal Seminal Plasma Factor.—Table 2 presents data obtained when seminal plasma was heated at 55°C. for various lengths of time and then used for ovum culture. The inhibitory effect of undiluted seminal plasma was not abolished by the heat treatment, but the cleavage percentages in the serum-heated plasma cultures suggest

some destruction by the heating of the inhibitory factor. This is emphasized by the complete failure of the ova to undergo cleavage in 50% serum-unheated plasma (data of table 1). In a subsidiary experiment, bull and rabbit seminal plasma were stored at 5°C. for 2 weeks and each was then mixed with equal volumes of blood serum. Cleavage of 75% of the ova cultured in this mixture suggests that the inhibitory factor is lost with time.

Rabbit ova were cultured at 38°C. or stored at 10°C. in rabbit seminal plasma or serum-seminal plasma mixtures for 1 day. Normal cleavage was not observed. Then the ova were cultured in undiluted serum for 1 day at 38°C. Table 3 presents the results. It is clear that the seminal plasma effect is most pronounced at body temperature. Since ovum

TABLE 2

CLEAVAGE OF RABBIT OVA CULTURED IN RABBIT SEMINAL PLASMA HEATED FOR VARIOUS LENGTHS OF TIME

SEMINAL PLASMA HEATED AT 55°C., MINUTES	DILUTED WITH RABBIT SERUM %	TOTAL NO. OF OVA	NO. SHOWING CLEAVAGE	CLEAVAGE, %
30	0	18	0	0
	50	26	6	23
120	0	15	0	0
	50	40	20	50

TABLE 3

CLEAVAGE OF RABBIT OVA CULTURED IN SERUM FOLLOWING CULTURE IN SEMINAL PLASMA FOR 1 DAY

TREATMENT BEFORE CULTURE		CLEAVAGE IN UNDILUTED SERUM AT 38°		
TEMP., °C.	DILUTED WITH SERUM, %	TOTAL NO. OF OVA	NO. SHOWING CLEAVAGE	CLEAVAGE, %
38	0	12	0	0
	50	21	4	19
10	0	19	5	26
	50	21	13	62

disintegration occurred only after the elapse of 80 minutes in undiluted seminal plasma at 30–38°C., the harmful effect involves either a slow process or the accumulation of metabolic products of the seminal plasma.[6] The ovocidal substance in diluted form may have inhibitory action because no ova cleaved in seminal plasma containing 50% serum at 38°C., but there was 19% cleavage when the ova were subsequently cultured in undiluted serum (table 3).

The ultrafiltrate of bull or rabbit seminal plasma diluted with 50% of serum does no obvious harm to the ova, suggesting that the harmful substance is of large molecular size. Seminal plasma contains about 10 times the amount of phosphorus contained in serum.[4] The high concentration of phosphorus compounds derived from the seminal vesicles[3] and the large

amount of phosphatase originating in the prostate glands[4] lead one to suspect that the harmful substance may be a heat labile, organic phosphocompound, emanating from these glands or derived from the metabolism of seminal plasma.

Rabbit blastocysts, recovered from the uterus at either 4 or 6 days after mating, were cultured in rabbit seminal plasma containing 50% of serum at 38°C. for 1 day, but no obvious evidence of damage was observed and ovum growth approximated that observed in control cultures. In contrast, it is interesting to note that 4- to 6-day rabbit blastocysts in the normal uterine environment are extraordinarily dependent on a hormone-labile nutrition, since ovariectomy[5] or estrogen administration[6] prevents their growth and development *in vivo*. Early rabbit ova, on the other hand, exhibit normal cleavage under these conditions which affect the blastocysts so adversely.[7] It is evident therefore that at their various developmental stages rabbit ova exhibit markedly different susceptibilities to factors affecting growth and development.

Summary.—Human, rabbit and bull seminal plasma causes the disintegration of fertilized rabbit ova and inhibits cleavage even when diluted with rabbit serum. The toxic factor appears to be most concentrated in bull seminal plasma and least so in human seminal plasma. The toxic factor is thermolabile, and disappears during prolonged storage at 5°C. Its typical action occurs after a measurable latent period which increases with increasing temperature.

* This investigation was supported by a grant from the Committee on Human Reproduction, National Research Council, acting on behalf of the National Committee on Maternal Health. Thanks are due to Dr. G. Pincus for encouragement during this study.

[1] Chang, M. C., *J. Gen. Physiol.*, **32**, 291 (1949).
[2] Corner, G. W., *Am. J. Physiol.*, **86**, 74 (1928).
[3] Huggins, C. M., and Johnson, A. A., *ibid.*, **103**, 574 (1933).
[4] Lundquist, F., *Nature*, **158**, 710 (1946).
[5] Mann, T., *Biochem. J.*, **39**, 451 (1945).
[6] Pincus, G., *Cold Spring Harbor Symposia of Quantitative Biology*, **5**, 44 (1937).
[7] Pincus, G., and Werthessen, N. T., *J. Exp. Zool.*, **78**, 1 (1938).

ABSOLUTE AND UNCONDITIONAL CONVERGENCE IN NORMED LINEAR SPACES

BY A. DVORETZKY AND C. A. ROGERS

INSTITUTE FOR ADVANCED STUDY, PRINCETON, N. J.

Communicated by Marston Morse, January 30, 1950

1. Let B be a real Banach space and denote by $||x||$ the norm of an element x of B. The series

$$\sum_{\nu=1}^{\infty} x_\nu \quad (x_\nu \in B, \nu = 1, 2, \dots) \tag{1}$$

is called absolutely convergent if $\sum ||x_\nu|| < \infty$; it is called unconditionally convergent if the series $\sum y_\nu$ converges whenever the sequence $(y_\nu)_1^\infty$ is a rearrangement of the sequence $(x_\nu)_1^\infty$. An equivalent definition of unconditional convergence of (1) is obtained by requiring $\sum \pm x_\nu$ to be convergent for every choice of the signs. There are several other equivalent definitions; most of these have been discussed by T. H. Hildebrandt.[1]

It is clear that if B is of finite (linear) dimension then (1) is unconditionally convergent if and only if it is absolutely convergent. The problem of finding the spaces for which these two types of convergence are equivalent is mentioned by S. Banach.[2] The primary aim of this note is to settle this problem by proving the following result.

THEOREM 1. *The unconditionally convergent series coincide with the absolutely convergent series if and only if the space B is of finite dimension.*

Here the only non-trivial assertion is that, if B is of infinite dimension, there is a series (1), which is unconditionally but not absolutely convergent. It is easy to give examples of such series in Hilbert space and similar examples have been given[3] for all the usually encountered infinitely dimensional Banach spaces. Interesting partial results on the problem solved by Theorem 1 have been established by M. E. Munroe[4] and S. Karlin.[5] The two last mentioned papers treat also some related problems and give various consequences of Theorem 1.

Our method of proof yields not only Theorem 1 but also the following result.

THEOREM 2. *If B is of infinite dimension and $\sum c_\nu$ is any convergent series of positive terms, then there exists an unconditionally convergent series (1) satisfying $||x_\nu||^2 = c_\nu$ for $\nu = 1, 2, \dots$.*

Applying this result with $c_\nu = \nu^{-1}[\log(1+\nu)]^{-2}$ we obtain:

COROLLARY: *If B is of infinite dimension then there exists an unconditionally convergent series (1) having the property that $\sum ||x_\nu||^{2-\epsilon} = \infty$ for every $\epsilon > 0$.*

Theorem 1 is obviously an immediate consequence of this Corollary.

If B is a Hilbert space then $\sum \|x_n\|^2 < \infty$ for every unconditionally convergent series (1). Thus Theorem 2 and its Corollary are in a certain sense best possible results.

A result (Lemma 1) concerning convex bodies in Euclidean space is proved in section 2. In section 3 this lemma is used to prove Theorem 2, and remarks are made concerning its extension. In section 4 some geometrical properties of convex bodies are obtained from Lemma 1 and from the construction used to prove this lemma.

2. We consider the n-dimensional Euclidean space of points $U = (u_1, \ldots, u_n)$ and use the usual vector notation. We first prove our main lemma.

LEMMA 1. *Let C be a body[6] which is convex and has the origin O as center, and let r be an integer with $1 \leq r \leq n$. Then there are n points $A_1, \ldots, A_n$ on the boundary of C such that, if $\lambda_1, \ldots, \lambda_r$ are any r real numbers with $1 \leq r \leq n$, then the point $\lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_r A_r$ is in the body λC where*

$$\lambda^2 = \left[2 + \frac{r(r-1)}{n}\right](\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_r^2). \tag{2}$$

Proof: We inscribe in C an ellipsoid with O as center having the largest possible n-dimensional volume. Since it is enough to establish the lemma for any affine transform of C, we may assume that this ellipsoid is the sphere S of unit radius.

We first show that after a suitable orthogonal transformation has been applied there will be r points $A_1, \ldots, A_r$ of contact of C with S, satisfying for $\rho = 1, 2, \ldots, r$

$$\left.\begin{aligned} A_\rho &= (a_{\rho 1}, a_{\rho 2}, \ldots, a_{\rho\rho}, 0, \ldots, 0), \\ a_{\rho 1}^2 + \ldots + a_{\rho(\rho-1)}^2 &= 1 - a_{\rho\rho}^2 \leq \frac{\rho - 1}{n}. \end{aligned}\right\} \tag{3}$$

For $r = 1$ this is clear; assuming it for $r = m - 1 < n$ we prove it for $r = m$. The ellipsoid

$$(1+\epsilon)^{n-m+1}(u_1^2 + \ldots + u_{m-1}^2) + (1 + \epsilon + \epsilon^2)^{-m+1}(u_m^2 + \ldots + u_n^2) \leq 1, \qquad (\epsilon > 0) \tag{4}$$

has a volume larger than that of S. Hence there is a point $A = A(\epsilon) = (a_1, \ldots, a_n)$ on the boundary of C in the ellipsoid (4). But, since A being on the boundary of C is not inside the unit sphere, we have $a_1^2 + \ldots + a_n^2 \geq 1$. It follows that A satisfies

$$[(1+\epsilon)^{n-m+1} - 1](a_1^2 + \ldots + a_{m-1}^2) + [(1 + \epsilon + \epsilon^2)^{-m+1} - 1](a_m^2 + \ldots + a_n^2) \leq 0. \tag{5}$$

If $\epsilon \to 0$ through a suitable sequence of positive numbers the corresponding sequence $A(\epsilon)$ will converge to a point A_m. It is clear from (4) that A_m is

a point of contact of S and the boundary of C, while from (5) we have in the limit

$$(n - m + 1)(a_{m1}^2 + \ldots + a_{m(m-1)}^2) + (-m + 1)(a_{mm}^2 + \ldots + a_{mn}^2) \leq 0. \quad (6)$$

By a suitable orthogonal transformation of the variables $u_m, \ldots, u_n$, leaving the points $A_1, \ldots, A_{m-1}$ invariant we may make the last $n - m$ coordinates of A_m vanish. Then, using (6) and the equation $a_{m1}^2 + \ldots + a_{mn}^2 = 1$ we obtain (3) with $\rho = m$. Thus (3) is proved for $\rho = 1, 2, \ldots, n$

Let $\lambda_1, \ldots, \lambda_r$ be any real numbers. By (3) the square of the distance from O to the point $\lambda_1 A_1 + \ldots + \lambda_r A_r$ is

$$\sum_{\sigma=1}^{r}\left(\sum_{\rho=\sigma}^{r}\lambda_\rho a_{\rho\sigma}\right)^2 \leq \sum_{\sigma=1}^{r}\left[2\lambda_\sigma^2 a_{\sigma\sigma}^2 + 2\left(\sum_{\rho=\sigma+1}^{r}\lambda_\rho a_{\rho\sigma}\right)^2\right]$$

$$\leq \sum_{\sigma=1}^{r} 2\left[\lambda_\sigma^2 a_{\sigma\sigma}^2 + \left(\sum_{\rho=\sigma+1}^{r}\lambda_\rho^2\right)\left(\sum_{\tau=\sigma+1}^{r}\ \sum_{\text{min. } (\rho-1,\,\tau-1)} a_{\tau\sigma}^2\right)\right]$$

$$= 2\sum_{\rho=1}^{r}\left[a_{\rho\rho}^2 + \sum_{\tau=1}^{r}\ \sum_{\sigma=1} a_{\tau\sigma}^2\right]\lambda_\rho^2.$$

But by (3), the last expression is less than or equal to

$$2\sum_{\rho=1}^{r}\left(1 + \sum_{\tau=1}^{r}\frac{\tau-1}{n}\right)\lambda_\rho^2 = \left[2 + \frac{r(r-1)}{n}\right]\sum_{\rho=1}^{r}\lambda_\rho^2 = \lambda^2.$$

Thus the point $\lambda_1 A_1 + \ldots + \lambda_r A_r$ is contained in the sphere λS and so is contained in the body λC. This proves the lemma.

3. Before we prove Theorem 2 it is convenient to obtain the following consequence of Lemma 1.

LEMMA 2. *Let B be a Banach space of infinite dimension and let $c_1, \ldots, c_r$ be any given positive numbers. Then there exist points $x_1, \ldots, x_r$ in B with $\|x_\rho\|^2 = c_\rho$ for $\rho = 1, \ldots, r$ and such that, if $\sum'$ denotes the sum over any subset of the numbers $1, \ldots, r$, then*

$$\left\|\sum' x_\rho\right\|^2 \leq 3\sum' c_\rho. \quad (7)$$

Proof: Write $n = r(r-1)$. As B is of infinite dimension we can choose n linearly independent elements $z_1, \ldots, z_n$. Then the points $U = (u_1, \ldots, u_n)$ with $\|u_1 z_1 + \ldots + u_n z_n\| \leq 1$ form a convex body C with the origin as center in n-dimensional Euclidean space. Let $A_1, \ldots, A_r$ be the points given by Lemma 1. Writing $A_\rho = (a_{\rho 1}, \ldots, a_{\rho n})$, we put

$$x_\rho = c_\rho^{1/2}(a_{\rho 1}z_1 + \ldots + a_{\rho n}z_n), \qquad \rho = 1, \ldots, r.$$

Then, as $A_1, \ldots, A_r$ are on the boundary of C, we have $\|x_\rho\|^2 = c_\rho$ for $\rho = 1, \ldots, r$. Further, as the point $\sum' c_\rho^{1/2} A_\rho$ is in λC where $\lambda^2 = 3\sum' c_\rho$, it follows that (7) is satisfied. This proves the lemma.

Proof of Theorem 2: Choose a strictly increasing sequence $n_1 = 0, n_2, n_3, \ldots$ of integers such that the series

$$\sum_{r=1}^{\infty}\left(\sum_{\nu=n_r+1}^{n_{r+1}} c_\nu\right)^{1/2}$$

is convergent. By Lemma 2 we can choose x_ν for $n_r < \nu \leq n_{r+1}$ so that $\|x_\nu\|^2 = c_\nu$ and $\left\|\sum^{(r)} x_\nu\right\|^2 \leq 3\sum^{(r)} c_\nu$, the sum $\sum^{(r)}$ being taken over any subset of the integers ν with $n_r < \nu \leq n_{r+1}$. Let $\sum y_\nu$ be any rearrangement of the series $\sum x_\nu$. Let $\epsilon > 0$ be given. Choose r so large that

$$\sum_{\rho=r}^{\infty}\left(\sum_{\nu=n_\rho+1}^{n_{\rho+1}} c_\nu\right)^{1/2} < \frac{\epsilon}{2}.$$

Choose p so large that the sum $\sum_{\nu<p} y_\nu$ includes all the terms x_ν with $\nu \leq n_r$. Then for any $q > p$ we have

$$\left\|\sum_{\nu=p}^{q} y_\nu\right\| \leq \sum_{\rho=r}^{\infty}\left\|\sum^{(\rho)} x_\nu\right\| \leq \sum_{\rho=r}^{\infty}\left(3\sum_{\nu=n_\rho+1}^{n_{\rho+1}} c_\nu\right)^{1/2} < \epsilon.$$

Since B is complete it follows that $\sum y_\nu$ is convergent. As this is true for every rearrangement of $\sum x_\nu$, the series $\sum x_\nu$ is unconditionally convergent and Theorem 2 is proved.

We note that the completeness of B was used only to deduce the convergence of $\sum y_\nu$ from its Cauchy convergence. Hence we have (with obvious meaning of unconditional Cauchy convergence)

THEOREM 3. *Let N be an infinitely dimensional normed linear space over the reals and $\sum c_\nu$ be any convergent series of positive numbers. Then there exists an unconditionally Cauchy convergent series $\sum x_\nu$ of elements of N satisfying $\|x_\nu\|^2 = c_\nu (\nu = 1,2, \ldots)$. In particular there exist such series with $\sum \|x_\nu\| = \infty$.*

Since a complex Banach space contains a real one, it is clear that Theorems 1 and 2 hold for complex Banach spaces. A similar remark applies to Theorem 3.

4. In this section we prove some geometrical results. The first result shows that Lemma 1 can be considerably improved in the special case where $r = n$ and $\lambda_1 = \pm 1, \ldots, \lambda_n = \pm 1$.

THEOREM 4. *Let C be a convex body with the origin O as center. Then there are points $P_1, \ldots, P_n$ on the boundary of C such that all the 2^n points $\pm P_1 \pm \ldots \pm P_n$ are in the body $2n^{1/2}C$.*

Proof: For $n > 1$ let q, r, s be the non-negative integers defined by

$$r(r-1) \leq 2n < r(r+1), \qquad n = qr + s, \qquad s < r. \quad (8)$$

Let $A_1, \ldots, A_r$ be the points thus denoted in Lemma 1 and for $t = 1, 2, \ldots, n$ put $P_t = A_{\nu(t)}$, where $\nu(t) \equiv t \pmod r$ and $1 \leq \nu(t) \leq r$.

Then all 2^n points $\pm P_1 \pm \ldots \pm P_n$ are of the form $\lambda_1 A_1 + \ldots + \lambda_r A_r$ where the integers λ_ν satisfy the inequalities

$$|\lambda_\nu| \leq q + 1 \text{ for } 1 \leq \nu \leq s, \qquad |\lambda_\nu| \leq q \text{ for } s < \nu \leq r.$$

Hence, by Lemma 1 all the 2^n points considered are in the body μC where

$$\mu^2 = \left[2 + \frac{r(r-1)}{n}\right] \sum_{\nu=1}^{r} \lambda_\nu^2 \leq 4\,[s(q+1)^2 + (r-s)q^2].$$

Taking account of (8) it is easily checked that $s(q+1)^2 + (r-s)q^2 < n^{3/2}$ for $n > 1$. The theorem being obvious for $n = 1$, is thus completely proved.

Remark: It is of some interest to find the exact dependence of μ on n. Our method, though capable of improving the constant 2 in this theorem, cannot improve the power in the estimate $\mu < 2n^{3/2}$. When C is a sphere then an enlargement by the factor $n^{1/2}$ is sufficient. Perhaps this is generally true, but we cannot prove it for $n \geq 3$.

We give a proof for $n = 2$ in the hope that it may be generalized to other values of n. Let B be the two-dimensional Banach space whose unit sphere is C. Given any point P_1 in this space with $\|P_1\| = 1$ there exists, by continuity, a point P_2 satisfying $\|P_2\| = 1$ and $\|P_1 + P_2\| = \|P_1 - P_2\|$. Let α denote this common norm, then also $\|\pm P_1 \pm P_2\| = \alpha$. Now put $Q_1 = (P_1 + P_2)/\alpha$, $Q_2 = (P_1 - P_2)/\alpha$, then $\|Q_1\| = \|Q_2\| = 1$ and $\|\pm Q_1 \pm Q_2\| = 2/\alpha$. Since $\min.(\alpha, 2/\alpha) \leq 2^{1/2}$ the proof is completed.

The following results are simple consequences of the construction used in proving Lemma 1. We include them since they seem to be of some geometrical interest.

THEOREM 5A. *Let C be a convex body with the origin as center. Then there is an ellipsoid $\mathcal{E}$ contained in C and a parallelopiped $\mathcal{P}$ containing C with volumes $V(\mathcal{E})$ and $V(\mathcal{P})$ satisfying*

$$\frac{V(\mathcal{P})}{V(\mathcal{E})} \leq \frac{2^n}{J_n}\left(\frac{n^n}{n!}\right)^{1/2}, \tag{9}$$

where J_n is the volume of the unit n-dimensional sphere.

Proof: Take $\mathcal{E}$ to be an ellipsoid with O as center having the largest possible volume. As in the proof of Lemma 1, we may suppose without loss of generality that $\mathcal{E}$ is the unit sphere S and denote by $A_1, \ldots, A_n$ points of contact of C and S satisfying (3). As C contains S the only tac-plane to C at A_r is the plane $a_{r1}u_1 + \ldots + a_{rr}u_r = 1$. Thus C is contained in the parallelopiped $\mathcal{P}$ defined by $|a_{r1}u_1 + \ldots + a_{rr}u_r| \leq 1$, $r = 1, 2, \ldots, n$. By (3) the volume of $\mathcal{P}$ satisfies

$$V(\mathcal{P}) = 2^n\,|a_{11}a_{22}\ldots a_{nn}|^{-1} \leq 2^n\left(\frac{n^n}{n!}\right)^{1/2} = 2^n\left(\frac{n^n}{n!}\right)^{1/2}\frac{V(\mathcal{E})}{J_n}.$$

THEOREM 5B. *Under the conditions of Theorem 5A there is an ellipsoid* $\mathcal{E}$ *containing C and an "octahedron"* $\mathcal{O}$ *contained in C, with*

$$\frac{V(\mathcal{E})}{V(\mathcal{O})} \leq (1/2)^n J_n (n!n^n)^{1/2}. \tag{10}$$

Proof: The result follows immediately by application of Lemma 3 to the body K which is the polar reciprocal of C.

THEOREM 6. *Let C and K be convex bodies with the origin as center, which are polar reciprocal. Then their volumes satisfy*

$$\frac{2^n J_n}{(n!n^n)^{1/2}} \leq V(C) \cdot V(K) \leq 2^n J_n \left(\frac{n^n}{n!}\right)^{1/2}. \tag{11}$$

Proof: By Lemma 3 we may suppose without loss of generality that C contains the unit sphere S and is contained in a parallelopiped $\mathcal{P}$ with volume $V(\mathcal{P})$ satisfying

$$V(\mathcal{P}) \leq 2^n \left(\frac{n_n}{n!}\right)^{1/2}. \tag{12}$$

Then K is contained in S and contains an "octahedron" $\mathcal{O}$ with

$$V(\mathcal{O}) \geq \frac{2}{(n!n^n)^{1/2}} \tag{13}$$

The inequalities (11) now follow trivially from (12), (13) and the inclusion relations $S \subset K \subset \mathcal{P}$ and $\mathcal{O} \subset K \subset S$.

The bounds on the right of (9) and (10) can be written in the form $(\gamma_n n)^{n/2}$ where γ_n tends to a positive limit as n tends to infinity. It is easy to see that it is impossible to obtain such bounds with γ_n tending to zero as n tends to infinity. The bounds in (11) are considerably closer than those obtained by K. Mahler[7] but they are probably very far from the best possible.

[1] *Bull. Am. Math. Soc.*, **46**, 959–962 (1940).

[2] *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 240.

[3] E.g., Orlicz, W., *Stud. Math.*, **4**, 51–47 (1933); Macphail, M. S., *Bull. Am. Math. Soc.*, **53**, 121–123 (1947).

[4] *Duke Math. J.*, **13**, 351–365 (1946).

[5] *Ibid.*, **15**, 971–985 (1948).

[6] I.e., the closure of a bounded open set.

[7] *Časopis Pěst. Mat. Fys.*, **68**, 93–102 (1939).

THE REGION OF VALUES OF THE DERIVATIVE OF A SCHLICHT FUNCTION*

BY ARTHUR GRAD

OFFICE OF NAVAL RESEARCH, WASHINGTON, D. C.

Communicated by S. Lefschetz, December 10, 1949

1. We say that a function

$$f(z) = \sum_{\nu=1}^{\infty} a_\nu z^\nu$$

belongs to class S if it is regular and schlicht in $|z| < 1$ and $a_1 = 1$. The purpose of this note is to indicate how the variational methods developed by A. C. Schaeffer and D. C. Spencer[1, 2] can be used to determine the region of values covered by $f'(z_0)$, when $f(z)$ ranges over the class S, and z_0 is any fixed point in $|z| < 1$. A more detailed discussion will appear in reference 2. We denote the region by $R(z_0)$. Henceforth, without explicit statement to the contrary, $f(z)$ belongs to class S.

The region $R(z_0)$ is essentially the solution to a more general problem. Let

$$f(w) = w + a_2w^2 + a_3w^3 + \ldots$$

be regular and schlicht in a simply connected domain $\mathfrak{D}$ containing the origin, and let $w = w(z)$ be the function belonging to class S which maps the unit circle on $\mathfrak{D}$. If we write $f(w) = F(z)$, then $F(z)$ belongs to class S. Writing $w_0 = w(z_0)$, we have $F'(z_0) = f'(w_0)w'(z_0)$. Since $w'(z_0)$ depends only on $\mathfrak{D}$ and z_0, not on f, it follows that the region of values at the point w_0 of functions regular and schlicht in $\mathfrak{D}$ is simply a Euclidean magnification and rotation of $R(z_0)$, the magnification factor being $z'(w_0)$.

It suffices to consider the case where z_0 is real and positive, because $F(z) = e^{-i\theta}f(e^{i\theta}z)$ belongs to class S, and $F'(z) = f'(e^{i\theta}z)$. We shall therefore take $z_0 = r$, $0 \leq r < 1$. Before explicitly determining $R(r)$, various of its geometrical properties are readily established.

Since the class S is compact, $R(r)$ is closed. It is also bounded, the precise bounds being given by the well-known distortion and rotation theorems. $R(r)$ has two symmetries. It is symmetric with respect to the real axis, since $\overline{f(\bar{z})}$ belongs to class S. Making the transformation $z = (w + r)/(1 + rw)$ and then normalizing, $f(z)$ is transformed into a function $F(w)$, belonging to class S, for which

$$F'(-r)f'(r) = 1/(1 - r^2)^2.$$

Thus $R(r)$ is symmetric by inversion in the circle about the origin of radius $1/(1 - r^2)$.

Clearly $R(r)$ varies continuously with r, for if $f'(r)$ is any point of $R(r)$, then given ϵ, there exists a δ such that $|f'(r + \Delta r) - f'(r)| < \epsilon$ for $|\Delta r| < \delta$, so there is a point of $R(r + \Delta r)$ arbitrarily near any point of $R(r)$ for Δr sufficiently small. The region $R(r)$ expands with increasing r; that is, $R(r_1) \subset R(r_2)$ for $r_1 < r_2$. Writing $r_1 = \rho r_2$, $0 < \rho < 1$, let $g(z) = f(\rho z)/\rho$. Then $g(z)$ belongs to class S, and $g'(r_2) = f'(r_1)$. Thus any point $f'(r_1)$ of $R(r_1)$ is a point $g'(r_2)$ of $R(r_2)$.

If $f'(r)$ is an interior point of $R(r)$, then $f'(r)$ belongs to $R(r - \Delta r)$ provided that Δr is sufficiently small. Hence there is a function $g(z)$ of class S such that $g'(r - \Delta r) = f'(r)$; that is, $g'(\rho r) = f'(r)$, where $\rho = (r - \Delta r)/r$. Writing $h(z) = g(\rho z)/\rho$, we have $h'(r) = f'(r)$, where $h(z)$ is bounded in $|z| < 1$. Conversely, if a function $f(z)$, belonging to the point $f'(r)$ of $R(r)$, is bounded or satisfies the weaker condition that the map of $|z| < 1$ by $w = f(z)$ has an exterior point w_0, then we can find an $\epsilon > 0$ such that the circle $|w - w_0| < \epsilon$ is exterior to the map of the unit circle by $f(z)$. It follows that the function

$$g(z) = \frac{w_0^2}{w_0^2 - e^{2i\theta}\epsilon^2}\left[f(z) + \frac{e^{2i\theta}\epsilon^2}{f(z) - w_0}\right] + \frac{w_0 e^{2i\theta}\epsilon^2}{w_0^2 - e^{2i\theta}\epsilon^2}$$

belongs to class S, where θ is an arbitrary real number. Differentiating, we find that

$$g'(r) = \left\{1 + \frac{f(r)[f(r) - 2w_0]}{w_0^2[f(r) - w_0]^2}\, e^{2i\theta}\epsilon^2 + o(\epsilon^2)\right\} f'(r),$$

which implies that a complete neighborhood of $f'(r)$ belongs to $R(r)$, and $f'(r)$ is an interior point.

If $0 \leq \rho < 1$, the function $f(\rho z)/\rho$ belongs to the point $f'(\rho r)$ of $R(r)$. The point $f'(\rho r)$ is an interior point of $R(r)$, since $f(\rho z)/\rho$ is bounded, and it follows that $R(r)$ is a closed domain. We note that if θ is real, then the curve $f'(re^{i\theta})$ lies in $R(r)$. Hence at a point $f'(r)$ on the boundary of $R(r)$ where it has a tangent, $f''(r)$ defines a vector normal to the boundary. It will be shown later that the boundary of $R(r)$ has a tangent everywhere.

Although it will also be shown in the sequel that $R(r)$ is simply connected, the region is not schlicht for $r > 1/\sqrt{2}$. This is owing to the fact that $\arg f'(r)$, $\arg f'(0) = 0$, can exceed π in magnitude when $r > 1/\sqrt{2}$ For this reason it is better to consider the set of values at $z = r$ of that branch of $\log f'(z)$ which vanishes at $z = 0$. Since for any function $f(z)$ the derivative $f'(z)$ does not vanish in $|z| < 1$, we see that any branch of $\log f'(z)$ is single-valued in $|z| < 1$. Let $L(r)$ be the domain of values of the branch of $\log f'(z)$ at $z = r$ which vanishes at $z = 0$.

For convenience we refer to functions $f(z)$ for which $\log f'(r)$ lies on the boundary of $L(r)$ as boundary functions. To each point on the boundary

there corresponds a unique boundary function. Henceforth, without explicit statement to the contrary, we will interpret $\log f'(z)$ as that branch defined in $|z| < 1$ which vanishes at $z = 0$.

2. It was proved by Schaeffer and Spencer[3] that boundary functions must satisfy the differential equation

$$-\alpha e^{i(\varphi+\psi)}\left(\frac{z}{w}\frac{dw}{dz}\right)^2\frac{2w-\alpha}{(w-\alpha)^2} = e^{-i(\varphi+\psi)}\frac{(z-e^{i\psi})^2(z-\rho e^{i\varphi})(z-e^{i\varphi}/\rho)}{(z-r)^2(z-1/r)^2}, \tag{2.1}$$

where the right side is non-negative on $|z| = 1$, and

$$-e^{2i(\varphi+\psi)}(1-r^2)^2 = (r-e^{i\psi})^2(r-\rho e^{i\varphi})(r-e^{i\varphi}/\rho) \tag{2.2}$$

Here φ, ψ and ρ are real constants, $0 < \rho \leq 1$, and $\alpha = w(r)$. Integrating (2.1) and taking limits as $z \to 0$ and $z \to r$, we obtain

$$\log f'(r) = 2\log\frac{1-re^{-i\psi}}{1-r^2} + 2i\log\frac{s(r)+\rho}{s(r)-\rho} + e^{-i(\varphi+\psi)}\left\{\log\frac{(1-\rho)(\rho-|s(r)|^2)}{(1+\rho)(\rho+|s(r)|^2)} + 2\arg\frac{1+s(r)}{1-s(r)}\right\} + \pi, \tag{2.3}$$

where

$$s^2(z) = \frac{z-\rho e^{i\varphi}}{z-e^{i\varphi}/\rho}$$

The logarithms and argument are principal values, and $Im\ s(r) < 0$. An indeterminacy of $2m\pi + 2n\pi i$ has been resolved through the substitution of the Koebe functions, $f(z) = z/(1 \pm z)^2$, which are known to be boundary functions.

For $0 < r \leq 1/\sqrt{2}$, the region $L(r)$ is bounded by a closed curve given by (2.3). This is actually the equation of a curve, since there is but one real independent variable; indeed, for each ψ there are unique ρ and $e^{i\varphi}$ which satisfy (2.2). Any boundary function $w = f(z)$ maps $|z| < 1$ onto the exterior of a single analytic slit in the w-plane extending to $w = \infty$.

For $1/\sqrt{2} < r < 1$, the region $L(r)$ is bounded by two arcs alternating with two straight-line segments. The arcs are given by (2.3) for ψ in the intervals

$$\begin{aligned} -\pi/4 + \tau < \psi < \pi/4 - \tau, \\ \pi/4 + \tau < \psi < 3\pi/4 - \tau. \end{aligned} \tag{2.4}$$

Here $\tau = \tau(r) = \arccos 1/\sqrt{2}r$, $0 < \tau < \pi/4$. A boundary function corresponding to a point on these arcs maps $|z| < 1$ onto the exterior of a single analytic slit in the w-plane extending to $w = \infty$ as in the preceding

case. For ψ at the ends of the intervals (2.4), we obtain the remainder of the boundary of $L(r)$, consisting of the two straight-line segments of extreme imaginary part

$$C_0(r) \leq Re \log f'(r) \leq C_1(r),$$
$$Im \log f'(r) = \pm(\log \frac{r^2}{1-r^2} + \pi), \tag{2.5}$$

where

$$C_0(r) = -\log (r^2 + \sqrt{2r^2 - 1}) + 2r,$$
$$C_1(r) = -\log (r^2 - \sqrt{2r^2 - 1}) - 2r.$$

A boundary function corresponding to a point on these line segments has extreme imaginary part and maps $|z| < 1$ onto the exterior of a slit in the w-plane. The slit consists of a straight-line segment extending from a finite point to $w = \infty$ and two arcs which are symmetrical with respect to the straight-line segment and meet it at its finite end to form three equal angles. In the case of boundary functions corresponding to the end-points of the line segments (2.5), one of the two arcs degenerates into a point.

3. For fixed r, the boundary of $R(r)$ has been represented in parametric form, the boundary functions varying continuously with the parameter of the boundary. If $f(z)$ is a boundary function, then $F(z) = f(\rho z)/\rho$ belongs to class S for $0 \leq \rho \leq 1$. Letting ρ vary from 1 to 0, we obtain a homotopic deformation of the boundary of $R(r)$ into the point $F'(r) = 1$. Thus $R(r)$ is simply connected.

For $0 < r \leq 1/\sqrt{2}$ the boundary of $R(r)$ has been shown to be analytic, and for $1/\sqrt{2} < r < 1$, it has been shown to be analytic with the exception of four points. We will now show that even at these four points the boundary has a tangent. Consider the function

$$\Phi(z) = f\left(\frac{z+\eta}{1+\bar{\eta}z}\right),$$

where η is a small complex number. The function

$$F(z) = \frac{1}{\Phi'(0)}\,[\Phi(z) - \Phi(0)]$$

clearly belongs to class S. Now let $f(z)$ extremalize $Im[e^{-i\tau}f'(r)]$, where τ is some fixed real number. Upon differentiating $Im[e^{-i\tau}F'(r)]$ with respect to η, it readily follows that

$$e^{-i\tau}[f''(r) - 2a_2 f'(r)] + e^{i\tau}r[\overline{rf''(r) + 2f'(r)}] = 0. \tag{3.1}$$

Let $f(z)$ be the boundary function corresponding to one of the points where the straight-line segments (2.5) meet the arcs (2.3). If $f''(r) \neq 0$, the curve $f'(re^{i\theta})$, θ real, lies in $R(r)$ and has a tangent at the point in question. Hence the interior angle of the boundary at the point is not less than π. If $f''(r) = 0$, then by (3.1)

$$a_2 e^{-i\tau} f'(r) = re^{i\tau}\overline{f'(r)}. \tag{3.2}$$

Since $f'(r) \neq 0$, we see that $a_2 \neq 0$. If the internal angle were less than π, there would be infinitely many values of τ satisfying (3.2) for the same function $f(z)$, and so for the same values of a_2 and $f'(r)$. This is impossible. Hence the internal angle at the point is not less than π. However, the internal angle cannot exceed π, since $\arg f'(r)$ is maximal at the point. Therefore the internal angle is precisely equal to π.

* This paper represents work carried out under an Office of Naval Research Contract.

[1] Schaeffer, A. C., and Spencer, D. C., "A Variational Method in Conformal Mapping," *Duke Math. J.*, **14**, 949–966 (1947).

[2] Schaeffer, A. C., and Spencer, D. C., "Coefficient Regions for Schlicht Functions," to appear as an *American Mathematical Society Colloquium Publication*.

[3] Schaeffer, A. C., and Spencer, D. C., "A General Class of Problems in Conformal Mapping," these PROCEEDINGS, **33**, 185–189 (1947).

*A GENERALIZATION OF POISSON'S DISTRIBUTION FOR MARKOFF CHAINS**

BY B. O. KOOPMAN

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

Communicated by J. F. Ritt, January 24, 1950

1. Introduction.—Let $U_{n,k}$ ($n = 1, 2, \ldots$; $k = 1, \ldots, n$) be an infinite triangular array of variates (chance variables). If each row ($U_{n,1}$, $\ldots$, $U_{n,n}$) represents a set of n independent trials (i.e., $U_{n,k} = 1$ or 0 according as the kth trial succeeds or fails), and if $P_n(s)$ is the probability of exactly s successes in this set, the Poisson distribution of mean m, $P_n(s) \to P(s) = e^{-m}m^s/s!$, is obtained in the case in which the individual probabilities of success, $p_{n,k} = p(U_{n,k} = 1)$, become small while the expected total number of successes approach m, as the number n of trials increases indefinitely (we are using $p(A)$ and $p(A|B)$ for the probability of event A, and the probability of A given B; and E will be used for the expected value). More precisely, the equations

$$\lim_{n \to \infty} \max_{1 \leq k \leq n} p_{n,k} = 0, \tag{1.1}$$

$$\lim_{n \to \infty} E\left(\sum_{k=1}^{n} U_{n,k}\right) = \lim_{n \to \infty} \sum_{k=1}^{n} p_{n,k} = m \geqq 0 \qquad (1.2)$$

together form the necessary and sufficient condition that $P_n(s)$ approach the Poisson distribution.[1] The latter is also called the *Law of Small Numbers*.

The present note considers the case in which each set of trials $(U_{n,1}, \ldots, U_{n,n})$, instead of being independent, forms a simple Markoff chain of length n.

In this case, in addition to the absolute probabilities $p_{n,k}$, the transition probabilities $(k = 2, \ldots, n)$:

$$\left.\begin{aligned} a_{n,k} &= p(U_{n,k} = 1 \,|\, U_{n,k-1} = 1), & a'_{n,k} &= 1 - a_{n,k}, \\ b_{n,k} &= p(U_{n,k} = 1 \,|\, U_{n,k-1} = 0), & b'_{n,k} &= 1 - b_{n,k}, \end{aligned}\right\} \qquad (1.3)$$

play an essential role. They are connected by the classical difference equation $(k = 2, \ldots, n)$:

$$p_{n,k} = r_{n,k}\, p_{n,k-1} + b_{n,k}, \qquad (r_{n,k} = a_{n,k} - b_{n,k}). \qquad (1.4)$$

To seek a generalization of Poisson's distribution for this case, we might impose condition (1.1). But the following more liberal condition:

$$\lim_{n \to \infty} p_{n,1} = 0 \text{ } \textit{implies} \lim_{n \to \infty} \max_{1 \leqq k \leqq n} p_{n,k} = 0 \qquad (1.5)$$

would seem to be more appropriate. It corresponds to the view that the essential features of the mechanism by means of which the trials are made are expressed by its transition probabilities, whereas the absolute probabilities are more incidental and are largely determined by the situation in the first trial.

In the Markoff case (implying (1.4) or its generalization), we will say that any limiting distribution $P_n(s) \to P(s)$ obtained under conditions implying (1.2) and (1.5) is a *law of small numbers*.

We shall establish a law of small numbers in the case of stationary transitions ($a_{n,k}$, $b_{n,k}$ independent of k); its generating function is of a simple exponential form and the probabilities $P(s)$ are given in terms of Laguerre polynomials. The note concludes with a statement, without proof, of the theorem giving the distribution in the non-stationary case, and considers further generalizations.

2. Preliminaries on Markoff Chains.—Let $\sigma_{n,k}$ be the standard deviation of $U_{n,k}$, and $\rho_{n,j,k}$ the correlation coefficient of $U_{n,j}$, $U_{n,k}$ $(1 \leqq j \leqq k \leqq n)$. Then it is easily seen from the difference equations of type (1.4) that[2] $\rho_{n,j,k} = r_{n,j+1}^{(k-j)}\, \sigma_{n,j}/\sigma_{n,k}$. Therefore the coefficient of regression of $U_{n,k}$ on $U_{n,j}$, being $\rho_{n,j,k}\sigma_{n,k}/\sigma_{n,j}$, is $r_{n,j+1}^{(k-j)}$.

To find the generating function of $P_n(s)$,

$$\varphi_n(t) = \sum_{s=0}^{n} P_n(s)t^s, \tag{2.1}$$

we introduce $u_{n,k,s}$ the probability that there be exactly s successes in the trials $(U_{n,1}, \ldots, U_{n,k})$ $(0 \leqq s \leqq k \leqq n)$. If $u'_{n,k,s}$ is the probability of this event and also that success occur on the kth trial and if $u''_{n,k,s}$ is a similar probability with failure in the kth, we have

$$u_{n,k,s} = u'_{n,k,s} + u''_{n,k,s}, \qquad u'_{n,k,0} = 0, \qquad u''_{n,k,k} = 0.$$

Simple probability reasoning establishes the classical difference equations

$$u'_{n,k+1,s+1} = a_{n,k+1}\,u'_{n,k,s} + b_{n,k+1}\,u''_{n,k,s}$$

$$u''_{n,k+1,s} = a'_{n,k+1}\,u'_{n,k,s} + b'_{n,k+1}\,u''_{n,k,s}.$$

Introducing the generating functions

$$\varphi_{n,k}(t) = \sum_{s=0}^{k} u_{n,k,s}\,t^s, \qquad \varphi'_{n,k}(t) = \sum_{s=0}^{k} u'_{n,k,s}\,t^s,$$

$$\varphi''_{n,k}(t) = \sum_{s=0}^{k} u''_{n,k,s}\,t^s,$$

we evidently have

$$\varphi_{n,k}(t) = \varphi'_{n,k}(t) + \varphi''_{n,k}(t), \qquad \varphi_{n,n}(t) = \varphi_n(t).$$

Furthermore, the above difference equations lead at once to the following matrix recurrence equation

$$\Phi_{n,k+1} = \Phi_{n,k}W_{n,k+1} \qquad (1 \leqq k \leqq n-1),$$

where

$$\Phi_{n,k} = \left\|\varphi'_{n,k}(t)\ \varphi''_{n,k}(t)\right\|,$$

$$W_{n,k} = \begin{Vmatrix} ta_{n,k} & a'_{n,k} \\ tb_{n,k} & b'_{n,k} \end{Vmatrix}.$$

Combining these relations, we obtain the expression

$$\varphi_n(t) = \left\|tp_{n,1}\ q_{n,1}\right\| W_{n,2}W_{n,3}\ldots W_{n,n}J, \quad J = \begin{Vmatrix} 1 \\ 1 \end{Vmatrix}.$$

3. The Case of Stationary Transitions.—In this section we assume that the transition probabilities $a_{n,k}$, $b_{n,k}$ are *stationary*, implying the existence of the infinite sequences $\{a_i\}$, $\{b_i\}$, with

$$a_{n,k} = a_n, \qquad b_{n,k} = b_n, \qquad (a'_n = 1 - a_n, \qquad b'_n = 1 - b_n). \tag{3.1}$$

The condition (1.5) applied to (1.4) leads to $b_n \to 0$. We assume further that

$$\lim_{n \to \infty} a_n = a < 1, \qquad \lim_{n \to \infty} p_{n,1} = p_1, \qquad (q_1 = 1 - p_1). \quad (3.2)$$

Summing (1.4) for $k = 2, \ldots, n$, and applying these relations, we obtain (since $a = \lim r_n = r$)

$$\left.\begin{aligned} b_n &= h/n + \theta_n/n, \\ h &= m(1 - r) - p_1, \qquad \lim_{n \to \infty} \theta_n = 0. \end{aligned}\right\} \quad (3.3)$$

Since $a < 1$ we may fix $\tau > 1$ with $a\tau < 1$. Let T be the circle $|t| \leqq \tau$ in the complex t plane. We will show that $\varphi_n(t) \to \varphi(t)$ uniformly on T as $n \to \infty$. From this it will follow, by use of Cauchy's integral formulas for the coefficients of t^s in $\varphi_n(t)$ and $\varphi(t)$ as contour integrals about the circle $|t| = \tau$, that $P_n(s) \to P(s)$, generated by $\varphi(t)$.

Applying the formulas of this section to the results at the end of section 2, we see that, once the uniform limit on T of $W_{n,2}^{(n-1)} = W_n^{n-1}$ has been found, we shall have, uniformly on T,

$$\varphi_n(t) \to \varphi(t) = \|tp_1 q_1\| WJ, \qquad W_n^{n-1} \to W. \quad (3.4)$$

To find W, reduce W_n to diagonal form. Its characteristic roots are

$$\left.\begin{matrix} \lambda'_n \\ \lambda''_n \end{matrix}\right\} = \frac{1}{2}\{b'_n + ta_n \mp [(b'_n + ta_n)^2 - 4t\,r_n]^{1/2}\}; \quad (3.5)$$

and we have, in virtue of (3.1)–(3.3),

$$\lim_{n \to \infty} \lambda'_n = ta, \qquad \lim_{n \to \infty} \lambda''_n = 1. \quad (3.6)$$

Since for all $t \in T$, $|ta| < 1$, these roots are distinct once n is sufficiently large. Then $W_n = K_n \Lambda K_n^{-1}$, where

$$\Lambda_n = \begin{Vmatrix} \lambda'_n & 0 \\ 0 & \lambda''_n \end{Vmatrix}, \qquad K_n = \begin{Vmatrix} a'_n & a''_n \\ \lambda'_n - ta_n & \lambda''_n - ta_n \end{Vmatrix}. \quad (3.7)$$

The limits of K_n, K_n^{-1}, being uniform on T, we have but to find the uniform limit $\Lambda_n^{n-1} \to \Lambda$, obtaining

$$W = \begin{Vmatrix} a' & a' \\ 0 & 1 - at \end{Vmatrix} \Lambda \begin{Vmatrix} 1/a' & -1/(1 - at) \\ 0 & 1/(1 - at) \end{Vmatrix}. \quad (3.8)$$

In view of (3.6) and (3.7),

$$\Lambda = \lim_{n \to \infty} \begin{Vmatrix} (\lambda'_n)^{n-1} & 0 \\ 0 & (\lambda''_n)^{n-1} \end{Vmatrix} = \begin{Vmatrix} 0 & 0 \\ 0 & \lambda \end{Vmatrix}, \quad (3.9)$$

where $\lambda = \lim (\lambda''_n)^{n-1} = \lim (\lambda''_n)^n$ (all limits uniform on T)

Direct integration shows that

$$(1+z)^{1/v} = 1 + \frac{z}{2} - \frac{z^2}{4}\int_0^1 \frac{(1-v)\,dv}{(1+vz)^{1/v}}, \qquad (|z| < 1),$$

the path of integration being the real segment (0, 1). Applying this to (3.5), and making use of (3.2) and (3.3), we find that

$$\lambda_n'' = 1 - \frac{h(1-t)}{1-rt}\frac{1}{n} + \frac{\Omega_n(t)}{n}, \tag{3.10}$$

where $\Omega_n(t) \to 0$ uniformly on T as $n \to \infty$.

Integration along the real segment $(0 \leqq v \leqq 1)$ shows that

$$\log(1+z) = z - z^2\int_0^1 \frac{(1+v)\,dv}{(1+vz)^2}, \qquad (|z| < 1).$$

Applying this and (3.10) to $\log(\lambda''_n)^n$, we finally derive the limit, uniform on T:

$$(\lambda''_n)^n \to \lambda = \exp\left[-\frac{h(1-t)}{1-rt}\right]. \tag{3.11}$$

Combining (3.3), (3.4) (3.8), (3.9), (3.11), we have:

THEOREM I. *If a Markoff chain with stationary transitions satisfies the small number conditions (1.2), (1.5), and the limit conditions (3.2), then it leads to a law of small numbers $P_n(s) \to P(s)$, generated by*

$$\varphi(t) = \left[1 - \frac{(1-t)p_1}{1-rt}\right]\exp\frac{-m(1-r)(1-t) + (1-t)p_1}{1-rt}. \tag{3.12}$$

Explicit expressions for $P(s)$ are easily found in terms of the Laguerre polynomials, defined by the equation

$$\frac{1}{1-x}\exp\frac{-wx}{1-x} = \sum_{s=0}^{\infty} L_s(w)x^s.$$

On setting

$$x = rt, \qquad w = -[m(1-r) - p_1](1-r)/r,$$

we obtain

$$P(s) = e^{-m(1-r)+p_1}[q_1 L_s(w)r^s + (p_1 - r)L_{s-1}(w)r^{s-1}].$$

4. *The Case of Non-Stationary Transitions.*—We continue to assume the small number condition (1.5); but (1.2) must be extended to take the form that, for each $v = 0, 1, 2, \ldots,$

$$\lim_{n\to\infty}\sum_{k=1}^{n-v} E(U_{n,k}U_{n,k+v}) = R_v. \tag{4.1}$$

Evidently $R_0 = m$. Furthermore (3.1) and (3.2) are replaced by

$$a_{n,k} \to a_k \leqq a < 1 \text{ uniformly on } (2 \leqq k \leqq n) \text{ as } n \to \infty; \quad (4.2)$$

$$b_{n,k}\left[\sum_{k=2}^{n} b_{n,k}\right]^{-1} = c_k\left[\sum_{k=2}^{n} c_k\right]^{-1}(1 + \epsilon_{n,k}), \quad (4.3)$$

where $\epsilon_{n,k} \to 0$ uniformly as $n \to \infty$: (4.3) is a statement of the uniform approach of $b_{n,2}:b_{n,3}:\ \ldots$ to limiting proportions $c_2:c_3:\ \ldots$ (c_i not all zero). We then have

THEOREM II. *Under hypotheses (1.5), (4.1), (4.2), (4.3), and the assumption $p_{n,1} \to p_1$, there follows the conclusion that $P_n(s) \to P(s)$ generated by the function*

$$\varphi(t) = [1 - f(t)] \exp\{-(1-t)[m+(t-1)\psi(t)] + f(t)\}, \quad (4.4)$$

where

$$\psi(t) = \sum_{\nu=1}^{\infty} R_\nu t^{\nu-1}, \qquad f(t) = (1-t)\sum_{\nu=0}^{\infty} p_1 a_2^{(\nu)} t^\nu. \quad (4.5)$$

The proof depends on a certain theorem in analysis.[3] Similar methods have recently led the author to formally similar results in the case where each trial has many possible outcomes (e.g., corresponding to points on an N-space), one class of which corresponds to "success," the rest to "failure"; matrix and operator multiplication enter.

[*] An outline of the paper which will appear in full under the title, "A Law of Small Numbers in Markoff Chains," presented to the American Mathematical Society in its meeting in New York, February 26, 1949. The investigation was suggested by the theoretical (and unclassified) part of developements made in 1948 by the author in the course of his work with the Operations Evaluation Group of the Chief of Naval Operations, U. S. Navy. Any views contained in this work are the author's and not the Navy's.

[1] Koopman, B. O., "Necessary and Sufficient Conditions for Poisson's Distribution," *Bull. Am. Math. Soc.* (forthcoming).

[2] We shall agree to write

$A_{n,i}^{(0)} = 1, \qquad A_{n,i}^{(\lambda)} = A_{n,i}A_{n,i+1}\ldots A_{n,i+\lambda-1},$

whatever the sequence of symbols $A_{n,i}$ ($i = 1, 2, \ldots$) may denote. In the present case, $r_{n,i}^{(k-i)} = r_{n,i+1}r_{n,i+2}\ldots r_{n,k}$.

[3] Cf. Koopman, B. O., "Exponential Limiting Products in Banach Algebras" (forthcoming).

ALMOST PERIODIC INVARIANT VECTOR SETS IN A METRIC VECTOR SPACE

By Wilhelm Maak

Hamburg University

Communicated by Hermann Weyl, December 15, 1949

In this note it will be shown that the axioms on which H. Weyl, in a recent paper published under the above title,[1] has based his theory may be reduced by a somewhat different approach. In particular, we shall be able to dispose of the axiom IV the artificial character of which he himself deplores.

We start from the main theorem of the theory of almost periodic functions in a group (see reference 2), but we prefer to express it in this way: Given a group $\mathfrak{G}$ and a closed modul $\mathfrak{M}$ of almost periodic functions in $\mathfrak{G}$. If $\mathfrak{M}$ is invariant under the transformations $f(x) \to f(cx)$, c arbitrary in $\mathfrak{G}$, we call $\mathfrak{M}$ a left-modul. Then we have: *Given an arbitrary left-modul $\mathfrak{M}$ consisting of almost periodic functions, this modul is always the smallest closed modul, containing all (finite) irreducible left-moduls in $\mathfrak{M}$.* (See reference 4.)

A theorem like this is correct also for abstract almost periodic functions in $\mathfrak{G}$. A function $f(x)$ on a group is called an abstract function, if the values f of the function are elements of a vector space $\mathfrak{V}$ instead of complex numbers. The abstract vector space $\mathfrak{V}$ (possibly with non-denumerable dimensions) is a linear space with the complex numbers as left-operators. (If we assume that the elements of $\mathfrak{G}$ are right-operators, we have a vector space, satisfying the axiom I in reference 1.) Besides this we suppose the elements $\mathfrak{u} \in \mathfrak{V}$ to have a length $|\mathfrak{u}|$. We require $\mathfrak{V}$ to be closed in the sense of this length. [We see, that $\mathfrak{V}$ satisfies the axiom III in reference 1, but without the formula (1.3). Besides this we do not assume the invariance of $|\mathfrak{u}|$, because the transformation with elements of the group is not necessarily defined.] The notion "almost periodic function with values in $\mathfrak{V}$" is defined in the same manner as in the case where $\mathfrak{V}$ consists of complex numbers (compare reference 5). The word "left-modul," also, is to have the same meaning as before, properly adjusted to the new case. From the main theorem, given above, and from the fact, that every abstract almost periodic function $f(x)$ can be uniformly approximated with arbitrary accuracy by folded functions $\varphi \times f$ with complex almost periodic functions φ, we conclude the *main-theorem* about abstract almost periodic functions: *Given an arbitrary left-modul $\mathfrak{M}$, consisting of abstract almost periodic functions, this modul $\mathfrak{M}$ is always the smallest closed modul, containing all (finite) irreducible left-moduls in $\mathfrak{M}$.*

From now on we suppose, that $\mathfrak{B}$ has the elements of $\mathfrak{G}$ as right-operators and that the length $|\mathfrak{u}|$ is invariant under the transformations of $\mathfrak{G}$. [We see, that $\mathfrak{B}$ fulfills the axioms I and III of reference 1, but without the inequality (1, 3).] Then a vector $\mathfrak{u} \in \mathfrak{B}$ is called almost periodic (as in reference 1), if $f(x) = \mathfrak{u}x$ is an almost periodic function of x. Now we consider a closed, invariant modul $\mathfrak{U} \subseteq \mathfrak{B}$, consisting of almost periodic vectors $\mathfrak{u} \in \mathfrak{B}$ only. Then the set of all functions $f(x) = \mathfrak{u}x$ with $\mathfrak{u} \in \mathfrak{U}$ is a left-modul of almost periodic functions. If we apply the above-mentioned main-theorem concerning abstract almost periodic functions, we obtain the *approximation theorem: Given an arbitrary invariant closed modul* $\mathfrak{U}$, *consisting of almost periodic vectors* $\in \mathfrak{B}$, *this modul* $\mathfrak{U}$ *is always the smallest closed modul, containing all (finite) irreducible moduls in* $\mathfrak{B}$. This theorem is more general than Weyl's "theorem of strong approximation,"[1] which can be deduced from our theorem by mere specialization.

The Parseval equation is an immediate consequence of our approximation theorem. But now it is necessary to have a scalar product $(\mathfrak{u}, \mathfrak{v})$ of the vectors $\mathfrak{u}, \mathfrak{v} \in \mathfrak{B}$, which is invariant under the transformations of $\mathfrak{G}$. Moreover it must be assumed that $\sqrt{(\mathfrak{u}, \mathfrak{u})} = \|\mathfrak{u}\| \leqq |\mathfrak{u}|$. (This is the inequality (1.3) of the axiom III in reference 1. From now on we suppose that all assumptions of Weyl's axioms I, II and III are fulfilled.) $\mathfrak{U} \subseteq \mathfrak{B}$ being a closed invariant modul, consisting of almost periodic vectors, we examine all (finite) invariant irreducible moduls $\mathfrak{U}' \subseteq \mathfrak{U}$. Each gives rise to an irreducible representation $D'(x)$ of $\mathfrak{G}$. All $\mathfrak{U}'$ belonging in this sense to a representation $D'(x)$ equivalent to a fixed representation $D(x)$, we unite into a modul $\mathfrak{U}_D$. We suppose, that all moduls $\mathfrak{U}_D$ are of finite dimension. (Denumerable dimensions would be also admissible. The invariant sets, which are considered by Weyl, are of this kind.) Then we can find in every $\mathfrak{U}_D$ an orthogonal and normed basis $\mathfrak{e}_1^{(D)}, \ldots, \mathfrak{e}_{nD}^{(D)}$. If now D runs over a complete, not necessarily denumerable, system of inequivalent irreducible representations $\ldots, D^{(\rho)}(x), D^{(\mu)}(x), \ldots$ of $\mathfrak{G}$ we obtain an orthogonal and normed system of vectors $\ldots, \mathfrak{e}_1^{(\rho)}, \ldots, \mathfrak{e}_{n\rho}^{(\rho)}, \mathfrak{e}^{(\mu)}, \ldots, \mathfrak{e}_{n\mu}^{(\mu)}, \ldots$, in $\mathfrak{U}$. Making use of $\|\mathfrak{u}\| \leqq |\mathfrak{u}|$, we deduce from the approximation theorem, that every vector $\mathfrak{u} \in \mathfrak{U}$ can be approximated in the mean (that is, in the sense of $\|\mathfrak{u}\|$) with arbitrary accuracy by finite linear combinations of the $\mathfrak{e}_i^{(\rho)}$. For because of the approximation theorem, this is correct even in the strong case (that is, in the sense of $|\mathfrak{u}|$). In the usual manner one concludes from this the *Parseval equation: We define the Fourier-coefficients of an almost periodic vector* $\mathfrak{u} \in \mathfrak{U}$ *to be the numbers* $\alpha_i^{(\rho)} = (\mathfrak{u}, \mathfrak{e}_i^{(\rho)})$. *Then we have*

$$\|\mathfrak{u}\|^2 = \sum_{\rho} \sum_{i} \left| \alpha_i^{(\rho)} \right|^2.$$

A fuller treatment of the theory outlined above will appear in *Mathematische Annalen*.

[1] Weyl, H., "Almost Periodic Invariant Vector Sets in a Metric Vector Space," *Am. J. Math.*, **71**, 178–205 (1949).

[2] Neumann, J. v., "Almost Periodic Functions in a Group I," *Trans. A.M.S.*, **36**, 445–492 (1934).

[3] Bochner, S., and Neumann, J. v., "Almost Periodic Functions in a Group II," *Ibid.*, **37**, 21–50 (1935).

[4] Maak, W., "Moduln fastperiodischer Funktionen," *Abh. Math. Sem. Hamburg Univ.* **16**, 56–71 (1949).

[5] Maak, W., "Abstrakte fastperiodische Funktionen," *Ibid.*, **11**, 367–380 (1936).

ON MEMBRANES AND PLATES

By G. Szegö

Department of Mathematics, Stanford University

Communicated by J. L. Walsh, January 20, 1950

I. Introduction.—1. We deal with three closely related problems of the Calculus of Variations having some connection with the theory of elastic deformations.

Let D be an arbitrary domain in the plane bounded by a simple curve C. We denote the area-element of D by $d\sigma$. The admissible functions u are defined in D and satisfy certain boundary conditions on C. We define the positive numbers $\lambda_1, \lambda_2, \lambda_3$ as follows:

$$\lambda_1^2 = \min. \frac{\int_D \int |\operatorname{grad} u|^2 \, d\sigma}{\int_D \int u^2 \, d\sigma}, \qquad u = 0 \text{ on } C, \tag{a}$$

$$\lambda_2^4 = \min. \frac{\int_D \int (\nabla^2 u)^2 \, d\sigma}{\int_D \int u^2 \, d\sigma}, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } C, \tag{b}$$

$$\lambda_3^2 = \min. \frac{\int_D \int (\nabla^2 u)^2 \, d\sigma}{\int_D \int |\operatorname{grad} u|^2 \, d\sigma}, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } C. \tag{c}$$

Obviously, $\lambda_2^2 \geqq \lambda_1 \lambda_3$.

In (a) we allow only continuous functions which have piecewise continuous first derivatives. In (b) and (c) we allow only functions with continuous first derivatives. For the sake of simplicity we assume that C is an analytic curve.

2. Problems (a) and (b) are classical; the quantities λ_1 and λ_2 are the fundamental frequencies of a membrane with fixed boundary and of a clamped plate, respectively. Problem (c) occurs in the study of the buckling of plates.[1] All the three problems have a considerable literature for which we refer to the book of Weinstein.

As to (a) Lord Rayleigh has formulated the following conjecture which

was first proved by G. Faber:[3] *For all membranes of given area the circular one has the gravest fundamental tone (lowest fundamental frequency).*

Our purpose is to prove an analogous theorem for the problems (*b*) and (*c*), i.e., for the quantities λ_2 and λ_3. This can be done under a certain hypothesis concerning the functions u for which the minima in (*b*) and (*c*) are attained. The *hypothesis* is the following: *The functions u minimizing problems (b) and (c) are different from zero throughout the domain D, that is, the fundamental vibrations do not produce any nodal lines.*

3. We note the Euler-Lagrange differential equations associated with the minimum problems formulated above:

$$\nabla^2 u + \lambda_1^2 u = 0, \tag{a'}$$

$$\nabla^2 \nabla^2 u - \lambda_2^4 u = 0, \tag{b'}$$

$$\nabla^2 \nabla^2 u + \lambda_3^2 \nabla^2 u = 0. \tag{c'}$$

It is of interest to point out the minima in question for the special case of a circle. Denoting the radius of the circle by a we have

$$\lambda_1 = j/a, \qquad \lambda_2 = k/a, \qquad \lambda_3 = j'/a$$

where j, k, j' denote the smallest positive root of the Bessel functions

$$J_0(x), \qquad J_0(x)I_0'(x) - J_0'(x)I_0(x), \qquad J_0'(x),$$

respectively. We have

$$j = 2.405, \qquad k = 3.19, \qquad j' = 3.832. \tag{3}$$

Thus the principal result can be expressed as follows: *Let D be an arbitrary domain and let a be the radius of the circle of the same area as D. Then, under the hypothesis formulated above, we have the bounds*

$$\lambda_1 \geqq j/a, \qquad \lambda_2 \geqq k/a, \qquad \lambda_3 \geqq j'/a. \tag{4}$$

The first inequality is the content of the theorem of Rayleigh-Faber, the second and the third are proved in the present paper (under the hypothesis mentioned above). In all the three cases the proof is based on the process of symmetrization as it was in the proof of Faber. However we shall modify Faber's argument at a point which will be essential in dealing with the more difficult second and third problem.

II. Problem (a): Membrane with Fixed Boundary.—1. We denote by u the minimizing function of I (*a*) which is *known* to be different from zero in the interior of D. Hence, we can assume that $0 \leqq u \leqq 1$. We denote the level curve $u = \rho$ by C_ρ, $0 \leqq \rho \leqq 1$, so that $C_0 = C$ and C_1 coincides with the point(s) at which the maximum $u = 1$ is attained. The set C_ρ consists of a finite number of separated curves; for the sake of simplicity we assume that C_ρ is a single Jordan curve.

Let us denote by $A(\rho)$ the area of the domain inside of C_ρ. Thus $A(0) = A$ is the area of the given domain D, $A(1) = 0$. We define the quantity R by the equation $A(\rho) = \pi R^2$ so that R is a decreasing function of ρ; the maximum R_0 of R is the radius of the circle which has the same area as the given domain D.

2. We symmetrize the level curves C_ρ by replacing C_ρ by a circle of radius R about a fixed point, say the origin. The domain D is replaced then by a circular disk $\bar{D}$ of radius R_0. We define on $\bar{D}$ a function $\bar{u}$ by the condition that $\bar{u} = f(\rho)$ on the circle of radius R; the function f will be determined in such a way that the integral in the numerator of I (a) does not change in the transition from D to $\bar{D}$ and from u to $\bar{u}$. That is, if $\overline{d\sigma}$ is the area-element of $\bar{D}$,

$$\int_D \int |\operatorname{grad} u|^2 \, d\sigma = \int_{\bar{D}} \int |\operatorname{grad} \bar{u}|^2 \, \overline{d\sigma}. \tag{1}$$

On the other hand, we shall prove that

$$\int_D \int u^2 \, d\sigma \leqq \int_{\bar{D}} \int \bar{u}^2 \, \overline{d\sigma}. \tag{2}$$

Also the boundary condition $\bar{u} = 0$ will be satisfied. This yields the assertion immediately.

We observe that this argument differs from that of Faber. He defines the "symmetrized" function $\bar{u}$ by the condition $\bar{u} = \rho$ and shows that this process diminishes the integral in the numerator and leaves the integral in the denominator unchanged.

3. In what follows we use the notation

$$G = |\operatorname{grad} u| = \frac{d\rho}{dn} \tag{3}$$

where $d\rho > 0$ and dn is the piece of the normal of the level curve C_ρ between the level curves C_ρ and $C_{\rho+d\rho}$. The area of the ring-shaped domain between these curves is $A(\rho) - A(\rho + d\rho) = -A'(\rho)\, d\rho$. On the other hand, if ds is the arc-element of C_ρ, we find for the element $d\sigma$ of this area $d\sigma = ds \cdot dn = ds \cdot d\rho / G$ so that

$$-A'(\rho) = |A'(\rho)| = \int_{u=\rho} G^{-1} \, ds. \tag{4}$$

By Schwarz's inequality

$$\int_{u=\rho} G \, ds \int_{u=\rho} G^{-1} \, ds \geqq \left(\int_{u=\rho} ds\right)^2 = (L(\rho))^2 \tag{5}$$

where $L(\rho)$ is the length of C_ρ. Using the isoperimetric inequality $(L(\rho))^2 \geqq 4\pi A(\rho)$ we obtain the following important inequality for the first integral occurring in (5) which we denote also by $P(\rho)$, $\rho > 0$;

$$P(\rho) = \int_{u=\rho} G \, ds \geqq \frac{4\pi A(\rho)}{|A'(\rho)|}. \tag{6}$$

Obviously

$$\int_D\int |\operatorname{grad} u|^2 d\sigma = \int_D\int G^2 d\sigma = \int_0^1 \int_{u=\rho} G^2 ds \cdot \frac{d\rho}{G} = \int_0^1 P(\rho)d\rho. \quad (7)$$

4. Now we define the function $\tilde{u} = f(\rho)$ as follows:

$$f(\rho) = \int_0^\rho (P(t))^{1/2} \left(\frac{|A'(t)|}{4\pi A(t)}\right)^{1/2} dt, \qquad \rho > 0. \quad (8)$$

This integral exists since $P(\rho)$ [see (7)] and $A'(\rho)$ are integrable. Also it tends to zero with ρ so that $\tilde{u}$ satisfies the boundary condition. We have $A'(\rho)\, d\rho = 2\pi R\, dR$, hence

$$|\operatorname{grad} \tilde{u}|^2 = \left(\frac{d\tilde{u}}{dR}\right)^2 = (f'(\rho))^2 \left(\frac{d\rho}{dR}\right)^2 = (f'(\rho))^2 \frac{4\pi^2 R^2}{(A'(\rho))^2} = (f'(\rho))^2 \frac{4\pi A(\rho)}{(A'(\rho))^2} = \frac{P(\rho)}{|A'(\rho)|}. \quad (9)$$

The area between the circles of radii R and $R + dR$ is $2\pi R\, dR = |A'(\rho)|\, d\rho$ so that we find

$$\int_D\int |\operatorname{grad} \tilde{u}|^2 d\sigma = \int_0^1 P(\rho)d\rho \quad (10)$$

which proves indeed (1).

On the other hand we conclude from (8), by (6),

$$f(\rho) \geqq \int_0^\rho dt = \rho. \quad (11)$$

Hence

$$\int_0^1 \rho^2 |A'(\rho)|\, d\rho \leqq \int_0^1 (f(\rho))^2 |A'|(\rho)\, d\rho \quad (12)$$

which proves (2).

This establishes the assertion.

III. Problem (b): Clamped Plate.—1. We denote the minimizing function again by u and *assume* that $0 \leqq u \leqq 1$ holds throughout the domain D. Let $0 < \rho < 1$. We consider the open set $u < \rho$ which consists in general of several simply or multiply connected components $K_\rho, K_\rho', \ldots$. One of them, say K_ρ, has the curve C ($u = 0$) as part of its boundary. On the rest of the boundary of $u < \rho$ we have $u = \rho$. We denote by $A(\rho)$ the area of the complementary set characterized by the condition $u \geqq \rho$. The function $A(\rho)$ is continuous and monotonically decreasing; $A(0)$ is the area of the given domain D and $A(1) = 0$.

2. We write again $|\operatorname{grad} u| = G$. There is no need for any change in the argument of II 3; in particular II (4) and inequality (5) hold. Here $L(\rho)$ is the total length of the set of curves $u = \rho$. Now we apply

the isoperimetric inequality $(L(\rho))^2 \geqq 4\pi A(\rho)$. Obviously more than this is true: $L(\rho)$ can be replaced by the total length $L_1(\rho)$ of the curves which together with C bound K_ρ. Also $A(\rho)$ can be replaced by the area $A_1(\rho)$ bounded by the curves just mentioned. We have $L(\rho) \geqq L_1(\rho)$, $A(\rho) \leqq A_1(\rho)$.

Thus II (6) holds without any change.

3. Now we introduce the notation, $\rho > 0$,

$$\int_{u=\rho} (\nabla^2 u)^2 \frac{ds}{G} = Q(\rho) \tag{1}$$

so that

$$\iint (\nabla^2 u)^2 \, d\sigma = \int_0^1 \int_{u=\rho} (\nabla^2 u)^2 \frac{ds}{G} \, d\rho = \int_0^1 Q(\rho) d\rho. \tag{2}$$

To be sure, $Q(\rho)$ has no meaning for $\rho = 0$ but the integral (2) is finite. By Schwarz's inequality:

$$\int_{u=\rho} (\nabla^2 u)^2 \frac{ds}{G} \int_{u=\rho} \frac{ds}{G} \geqq \left(\int_{u=\rho} \nabla^2 u \frac{ds}{G} \right)^2 \tag{3}$$

or

$$(Q(\rho))^{1/2} \mid A'(\rho) \mid^{1/2} \geqq \int_{u=\rho} \nabla^2 u \frac{ds}{G}. \tag{4}$$

The integral of the function on the left-hand side of (4) exists in $0 \leqq \rho \leqq 1$ since $Q(\rho)$ and $A'(\rho)$ are integrable. Integrating we obtain

$$\int_0^\rho (Q(t))^{1/2} \mid A'(t) \mid^{1/2} dt \geqq \int_0^\rho \int_{u=t} \nabla^2 u \frac{ds}{G} \, dt =$$

$$\iint_{0 \leqq u \leqq \rho} \nabla^2 u \, d\sigma = - \int_{u=0,\ u=\rho} \frac{\partial u}{\partial n} ds = - \int_{u=\rho} \frac{\partial u}{\partial n} ds =$$

$$\int_{u=\rho} G \, ds = P(\rho) \geqq \frac{4\pi A(\rho)}{|A'(\rho)|}. \tag{5}$$

The integration indicated by the condition $u = \rho$ has to be extended over a set of curves $u = \rho$ and the normal is directed in each case into the interior of the domain $0 < u < \rho$.

4. We define now the function $f(\rho)$ in the following way. Let

$$g(\rho) = \frac{|A'(\rho)|}{4\pi A(\rho)} \int_0^\rho (Q(t))^{1/2} \mid A'(t) \mid^{1/2} dt, \ \rho > 0, \tag{6$_1$}$$

and

$$f(\rho) = \int_0^\rho g(\rho) d\rho, \qquad f(0) = 0. \tag{6$_2$}$$

The integral in (6_2) exists since $(0 < \rho_0 < \rho)$

$$\int_{\rho_0}^{\rho} -A'(\rho)d\rho \int_0^{\rho} (Q(t))^{1/2} (-A'(t))^{1/2}\, dt$$
$$= A(\rho_0) \int_0^{\rho_0} (Q(t))^{1/2} (-A'(t))^{1/2}\, dt - A(\rho) \int_0^{\rho} (Q(t))^{1/2} (-A'(t))^{1/2}\, dt$$
$$+ \int_{\rho_0}^{\rho} A(\rho)(Q(\rho))^{1/2} (-A'(\rho))^{1/2}\, d\rho.$$

In view of (5) we have $g(\rho) \geqq 1$, $f'(\rho) \geqq 1$ so that we obtain the important inequality

$$f(\rho) \geqq \rho, \qquad 0 \leqq \rho \leqq 1. \tag{7}$$

5. We determine $R = R(\rho)$ as in I, $A(0) = \pi R_0^2$ being the area of D. We consider the function $\tilde{u} = \tilde{u}(R) = f(\rho)$ defined on the circle $\bar{D}$ of radius R_0. This function satisfies the boundary conditions

$$\left.\begin{aligned} (\tilde{u})_{R=R_0} &= f(0) = 0, \\ \left(\frac{\partial \tilde{u}}{\partial n}\right)_{R=R_0} &= \lim_{R \to R_0} \tilde{u}'(R) = \lim_{\rho \to 0} f'(\rho) \frac{2\pi R}{|A'(\rho)|} = 0, \end{aligned}\right\} \tag{8}$$

in view of (6_1).

We have now

$$\nabla^2 \tilde{u} = \frac{1}{R}\frac{d}{dR}\left(R\frac{d\tilde{u}}{dR}\right) = \frac{4\pi}{A'(\rho)}\frac{d}{d\rho}\left(\frac{A(\rho)}{A'(\rho)} f'(\rho)\right) = \frac{(Q(\rho))^{1/2}}{|A'(\rho)|^{1/2}} \tag{9}$$

and we find

$$\int\!\!\int_{\bar{D}} (\nabla^2 \tilde{u})^2\, \overline{d\sigma} = \int_0^{R_0} (\nabla^2 \tilde{u})^2\, 2\pi R\, dR = \int_0^1 \frac{Q(\rho)}{|A'(\rho)|} |A'(\rho)|\, d\rho. \tag{10}$$

Comparing this with (2) we see that this process did not change the integral in the numerator of I (b).

On the other hand, the integral in the denominator will be for the circular domain $\bar{D}$:

$$\int_0^1 (f(\rho))^2 |A'(\rho)|\, d\rho \geqq \int_0^1 \rho^2 |A'(\rho)|\, d\rho. \tag{11}$$

This completes the proof.

IV. Problem (c): Buckling of a Plate.—1. Dealing with problem (c), the previous argument needs only slight modifications. We define $f(\rho)$ in the same way as in III so that the integral in the numerator of I (c) does not change.

As to the integral in the denominator we have by II (7):

$$\int_D\int |\operatorname{grad} u|^2\, d\sigma = \int_0^1 P(\rho)\, d\rho \tag{1}$$

On the other hand [cf. II (9)]

$$| \text{ grad } \bar{u} |^2 = (f'(\rho))^2 \frac{4\pi A(\rho)}{(A'(\rho))^2}, \tag{2}$$

and the area between the circles R and $R + dR$ is $| A'(\rho) | d\rho$ so that

$$\int_D \int | \text{ grad } \bar{u} |^2 \overline{d\sigma} = \int_0^1 (f'(\rho))^2 \frac{4\pi A(\rho)}{| A'(\rho) |} d\rho. \tag{3}$$

But

$$f'(\rho) \geq \frac{| A'(\rho) |}{4\pi A(\rho)} P(\rho) \geq 1. \tag{4}$$

so that

$$\int_0 \int |\text{grad } \bar{u} |^2 \overline{d\sigma} \geq \int_0^1 f'(\rho) \frac{4\pi A(\rho)}{| A'(\rho) |} d\rho \geq \int_0^1 P(\rho)\, d\rho = \int_D \int | \text{ grad } u |^2 d\sigma. \tag{5}$$

This establishes the assertion.

[1] A. Weinstein, "Étude des spectres des équations aux dérivées partielles de la théorie des plaques élastiques," *Mém. Sci. Math.*, vol. **88**, 1937, 62 pp.

[2] Lord Rayleigh, *The Theory of Sound*, 2nd ed., vol. 1, 1894, p. 345; Faber, G., "Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt," *Sitzber. Bayer. Akad.*, 1923, 169–172; independently, a proof was given by Krahn, E., "Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises," *Math. Ann.*, **94**, 97–100 (1924).

SPHERE-GEOMETRICAL UNITARY FIELD THEORY

By Tsurusaburo Takasu

Tohoku University, Sendai, Japan

Communicated by Oswald Veblen, November 17, 1949

A conformal relativity theory has long been longed for. Now I have arrived at the following results:

1. The Kaluza-Klein space[1,2] is equivalent to the Einstein space $V_4(R^{ij} = {}^1/_2 g^{ij} R)$ (special) dual-conformal (i.e., N.E.-Laguerre connection) geometrically so that the points in V_4 correspond to the generalized hyperspheres whose developments in the N.E. tangential spaces are hyperspheres of equal radii.

2. The Einstein-Mayer space[3] is equivalent to the Einstein space V_4 (special) Laguerre connection geometrically so that the points in V_4

correspond to the generalized hyperspheres whose developments in the Euclidean tangential spaces are hyperspheres of equal radii. (This fact was suggested by Y. Tomonaga.[13])

3. The Hoffmann space of the first kind,[4] which is a generalization of the Kaluza-Klein space, is equivalent to the Einstein space (special) Lie connection geometrically so that the points in the Einstein space correspond to the special linear hypercomplexes of generalized hyperspheres, whose developments in the N.E. tangential spaces are hyperspheres of equal radii.

4. The Hoffmann space of the second kind,[4] which is a generalization of the Einstein-Mayer space, is equivalent to the Einstein space (special) parabolic Lie connection geometrically so that the points in the Einstein space correspond to the special linear hypercomplexes of generalized hyperspheres, whose developments in the tangential Euclidean spaces are hyperspheres of equal radii. Thereby the parabolic Lie's geometry, which is quite new, is situated among others as follows:

(Dual-conformal geometry): (Laguerre's geometry)
= (Lie's geometry): (parabolic Lie's geometry).

In the two cases 4 and 5 the said hypercomplexes are of equal generalized radii.

Our results seem to answer the question of four-dimensionality of the unitary field theory completely and to be very suggestive for its expected future developments (e.g., for the case of that including the meson field).[17,18] The conformality long longed for seems to correspond to the similarity character of the Laguerre transformations in the wider (i.e., equiform) sense.

[1] Kaluza, Th., "Zum Unitätsproblem der Physik," *Sitz. preuss. Akad. Wiss.*, 966–972 (1921).

[2] Klein, O., "Quantentheorie und fünfdimensionale Relativitätstheorie," *Z. Physik*, **37**, 895–906 (1926).

[3] Einstein, A., and Mayer, W., "Einheitliche Theorie von Gravitation und Elektrizität," *Sitz. preuss. Akad. Wiss.*, 541–557 (1931). Part II, *Ibid.*, 130–137.

[4] Hoffmann, B., "A Generalization of the Kaluza-Klein Field Theory," *Quart. J. Math.*, **7**, 20–31 (1936).

[5] Hoffmann, B., "A Generalization of the Einstein-Mayer Field Theory," *Ibid.*, **7**, 32–42 (1936).

[6] Yano, K., "Sur la théorie unitaire non-holonome des champs. I," *Proc. Phys.-Math. Soc. Japan*, **19** (3), 867–896 (1937). II, *Ibid.*, 945–976.

[7] Yano, K., "La théorie unitaire des champs proposée par M. Vrancenau," *Compt. Rend. Paris*, **204**, 332–334 (1937).

[8] Yano, K., "Sur la nouvelle théorie unitaire de MM. Einstein et Bergmann," *Proc. Imp. Acad. Tokyo*, **14**, 325–328 (1938).

[9] Yano, K., "La relativité non-holonome et la théorie unitaire de MM. Einstein et Mayer," *Mathematica*, **14**, 121–132 (1938).

[10] Yano, K., "Sur la nouvelle théorie unitaire de MM. Einstein et Bergmann," *Proc. Imp. Acad. Tokyo*, **14**, 325–328 (1938).

[11] Einstein and Bergmann, "On a Generalization of Kaluza's Theory of Relativity," *Ann. Math.*, **36**, 683–701 (1938).

[12] Yano, K., "Unified Field Theory of Jordan-Bergmann" (in Japanese), *The Sugaku*, 1 (2), 91 (1944).

[13] Tomonaga, Y., "A Vector Field Based on a Riemannian Manifold" (in Japanese), *The Zenkoku-shijo-sugaku-danwakai*, **3** (1), 212–214 (1948).

[14] Schouten, J. A., and Haantjes, J., "Ueber die konforminvariante Gestalt der Maxwellschen Gleichungen und der elektromagnetischen Impulsenergie gleichungen," *Physica*, **1**, 869–872 (1934).

[15] Schouten, J. A., and Haantjes, J., "Ueber die konforminvariante Gestalt der relativistischen Bewegungsgleichungen, *Proc. Koninkl. Akad. Wetenschap. Amsterdam*, **39**, (5), 1–8 (1936).

[16] Veblen, O., *Projektive Relativitätstheorie*, Berlin, 1933.

[17] Hoffmann, B., "The Vector Meson Field and Projective Relativity," *Phys. Rev.*, **72** (2), 458–465 (1947).

[18] Hoffmann, B., "The Gravitational, Electromagnetic, and Vector Meson Fields and the Similarity Geometry," *Ibid.*, **73** (2), 30–35 (1948).

[19] Bergmann, P. G., "Unified Field Theory with Fifteen Field Variables," *Ann. Math.*, **49**, 255–264 (1948).

ERRATA

In the article "On the Interpretation of Multi-Hit Survival Curves," these PROCEEDINGS, pages 696–712, December, 1949, equation (21) on p. 702 should read

$$S = \prod_{i=1}^{m} [1 - (1 - e^{-k_i D})] \qquad (21)$$

and equation (22), same page, should read

$$S = [1 - (1 - e^{-kD})]^m \qquad (22)$$

K. C. ATWOOD

INFORMATION TO CONTRIBUTORS

THE PROCEEDINGS is the official organ of the NATIONAL ACADEMY OF SCIENCES and of the NATIONAL RESEARCH COUNCIL for the publication of brief accounts of important current researches of members of the ACADEMY and of the COUNCIL, and of other AMERICAN investigators. THE PROCEEDINGS will aim especially to secure prompt publication of original announcements of discoveries and wide circulation of the results of AMERICAN research among investigators in other countries and in all branches of science.

ARTICLES should be brief. The viewpoint should be comprehensive in giving the relation of the paper to previous publications of the author or of others and in exhibiting, where practicable, the significance of the work for other branches of science. Elaborate technical details of the work and long tables of data should be avoided, but authors should be precise in making clear the new results and should give some record of the methods and data upon which they are based.

MANUSCRIPTS should be prepared with a current number of the PROCEEDINGS as a model in matters of form, and should be typewritten in duplicate with double spacing, the author retaining one copy. Illustrations should be confined to text figures of simple character, though more elaborate illustration may be allowed in special instances to authors willing to pay for their preparation and insertion. Particular attention should be given to arranging tabular matter in a simple and concise manner.

REFERENCES to literature, numbered consecutively, will be placed at the end of the article and short footnotes should be avoided. It is suggested that references to periodicals be furnished in some detail and in general in accordance with the standard adopted for the Subject Catalogue of the International Catalog of Scientific Literature, viz., name of author, with initials following (ordinarily omitting title of paper), abbreviated name of Journal, with place of publication, series (if any), volume, inclusive pages, year. For example: Montgomery, T. H., *J. Morph., Boston*, 22, 131–815 (1911); or, Wheeler, W. M., *Koenigsberg. Schr. physik. Ges.*, 38, 1–142 (1914).

MANUSCRIPTS may be sent to the Managing Editor, Edwin Bidwell Wilson, Harvard School of Public Health, 665 Huntington Ave., Boston 15, Mass., only by members of the ACADEMY or by chairmen of the divisions of the NATIONAL RESEARCH COUNCIL who will assume responsibility for the propriety and scientific standards of the paper. Articles are limited to six printed pages; excess space is billed to the author at $5.00 per page.

PROOF will ordinarily be sent—with the understanding that charges for an author's corrections shall be billed to him. Authors are therefore requested to make final revisions on the typewritten manuscripts.

REPRINTS should be ordered at the time of submission of manuscript. They will be furnished to authors at cost, approximately as follows:

SCHEDULE OF RATES FOR REPRINTS, CARRIAGE CHARGES EXTRA

No. Copies	50	100	150	200	250
4 pp.	3.89	4.11	4.31	4.59	4.86
8 pp.	7.38	8.01	8.24	10.28	10.56
Cover	3.1[illegible]	3.60	4.35	5.05	5.85